

ECON0048 Lec 1,2 Revision Notes

Ambrose W

Lecture 1

1. Financial Market - Intro

Financial Assets

Definition: Claims to future resources; promises to pay a certain amount in the future. They differ in terms of (1) when and how much they pay, and (2) the possibility of default (credit risk) or inflation risk (if they deliver nominal returns).

Real Assets:

- If put in place today, they deliver an output of resources later.
- Real assets determine the productive capacity of an economy.

Classification of Financial Assets:

1. Equities: Claims to a share of the profit of a corporation forever.
2. Fixed-income securities (bonds, bills): Promise to make a fixed payment in the future.
3. Derivative securities: Make payments that depend on the prices of an underlying asset (e.g. futures, options, swaps, CDS).

Function of Capital Market:

- Allocate resources and risk efficiently across people.
- Shift purchasing power intertemporally.
- Shift risks across people and share the risk of uncertain opportunities.
- Allocate resources to the most productive investment.

Basic Concepts of Finance

- **Optimality [Portfolio Theory]:** What is the best combination of risky financial assets, given their returns and an individual investor's preferences?
- **Equilibrium:** What is the nature of asset prices and allocations when all investors have optimized and the market clears?
- **Arbitrage:** "No Arbitrage Assumption" means there are no risk-free profit opportunities.

2. Simple Notation for Asset Payoffs

Consider a finite number of assets, indexed by $n \in \{1, 2, \dots, N\}$, and a finite number of future states of the world, indexed by $s \in \{1, 2, \dots, S\}$. For each asset n and state s , let

$$X_{n,s}$$

denote the payoff of asset n in state s . These payoffs can be collected into an $N \times S$ matrix X , defined by:

$$X = \begin{pmatrix} X_{1,1} & X_{1,2} & \cdots & X_{1,S} \\ X_{2,1} & X_{2,2} & \cdots & X_{2,S} \\ \vdots & \vdots & \ddots & \vdots \\ X_{N,1} & X_{N,2} & \cdots & X_{N,S} \end{pmatrix}.$$

- **Row interpretation:** Each row of X corresponds to a particular asset. In other words,

$$(X_{n,1}, X_{n,2}, \dots, X_{n,S})$$

is the vector of payoffs for asset n across all possible states $s = 1, \dots, S$.

- **Column interpretation:** Each column of X corresponds to a particular state. In other words,

$$\begin{pmatrix} X_{1,s} \\ X_{2,s} \\ \vdots \\ X_{N,s} \end{pmatrix}$$

is the vector of payoffs (across different assets) in state s .

Prices

At time $t = 1$, each asset n is traded at some price p_n . These prices can be collected into a column vector

$$p = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix}.$$

Example: Riskless Bonds

A riskless bond might have the same payoff in every state:

$$X_{\text{bond},s} = X_{\text{bond},1} = X_{\text{bond},2} = \cdots = X_{\text{bond},S}.$$

If its price is p_f , then the (continuously compounded) riskless interest rate could be expressed as

$$\frac{1}{p_f} - 1,$$

depending on the exact convention of the model. This bond's constant payoff reflects no uncertainty across states.

3. Portfolio Choice

A portfolio can be established by buying w_n units of asset n for each $n \in \{1, \dots, N\}$.

$$w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}.$$

Price/Value of Portfolio

$$w^T p = \sum_{n=1}^N w_n p_n.$$

Payoff of the Entire Portfolio for State s

$$\sum_{n=1}^N w_n X_n^s.$$

Or in vector form,

$$w^T X = \left(\sum_{n=1}^N w_n X_n^1, \dots, \sum_{n=1}^N w_n X_n^S \right).$$

Note that the portfolio can be viewed as a newly constructed asset with its own payoff profile.

4. (State-Contingent) Arrow-Debreu Securities

Definition: A contract that pays one numeraire if a particular state s occurs and pays nothing otherwise (state price security).

Example: $AD_s = [0, \dots, 1, \dots, 0]$ (with the 1 in the s -th position).

Price of AD Security at State s : q^s , often called a *state price*. It captures how much it costs “now” to receive 1 in the future *only if* state s occurs.

5. Complete Market (Arrow-Debreu Market)

Definition: A market is complete if there are no transaction costs, perfect information, and every asset in every possible state of the world has a price.

If there are S Arrow-Debreu securities (one for each state) with the payoff matrix (an $S \times S$ identity matrix),

$$X_{AD} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

then any payoff profile $[X_n^1, \dots, X_n^S]$ can be obtained by combining Arrow-Debreu securities. In practice, these securities might be replicated if there are sufficiently many diverse assets, satisfying $N \geq S$ and full rank of the payoff matrix.

Relative Pricing: We can use different financial assets (possibly with negative weights for short-selling) to price new assets.

6. State-Contingent Market-Completeness Condition

Payoff Matrix (X)

We consider an $S \times N$ matrix (rows as states, columns as assets).

Replicating an AD Security

If we want to replicate $AD_s = [0, \dots, 1, \dots, 0]$, we look for a portfolio weight vector $w = (w_1, \dots, w_N)^T$ such that

$$w^T X = \left(\sum_{n=1}^N w_n X_n^1, \dots, \sum_{n=1}^N w_n X_n^S \right) = [0, \dots, 1, \dots, 0].$$

This amounts to solving S linear equations in N unknowns.

Market Completeness

The market is complete if there is a unique solution for every such replication problem, requiring:

- $N \geq S$,
- The columns of X are linearly independent (full column rank).

7. Arbitrage in Theory

Definition: An *arbitrage opportunity* is an investment strategy that:

- Never requires a cash outflow now or in the future, *and*
- Generates a cash inflow that is either:
 - *Type I Arbitrage:* in one or more future states but not now,
 - *Type II Arbitrage:* right now and possibly in one or more future states too.

No-arbitrage Assumption

Financial markets are assumed efficient enough that any arbitrage opportunities would be exploited and disappear quickly.

Implication: No-arbitrage gives strong restrictions on the relative prices of assets.

8. Law of One Price (LOOP)

Definition: Two assets or portfolios with the same payoffs in every future state must have the same price. If not, a Type II arbitrage emerges by buying the cheaper combination and shorting the more expensive one.

AD Securities Cannot Have Negative or Zero Price

- If $q^s < 0$, buy it to get immediate positive cash inflow and a future payoff, generating a Type II arbitrage.
- If $q^s = 0$, buy it to get a future payoff with no cost, a Type I arbitrage.

9. No-Arbitrage Under a Complete Market

If markets are complete, the price of any asset with payoff $[X_n^1, \dots, X_n^S]$ is given by

$$p_n = \sum_{s=1}^S X_n^s q^s,$$

where q^s are the state prices. By the Law of One Price, this formula must hold to prevent arbitrage.

Lecture 2

1. Measuring Return

Measures how well a security or asset performs as an investment over a given period, which may differ from its full maturity.

Let D_{t+1} be the payoff tomorrow (e.g. coupon or dividend) of an asset purchased today at P_t , which can be sold tomorrow at a price P_{t+1} .

Holding Period Return (HPR)

$$\text{HPR} = \frac{\text{Ending price} - \text{Initial price} + \text{Cash dividend}}{\text{Beginning Price}}.$$

Equivalently,

$$\text{HPR} = R_{t \rightarrow t+1} = \frac{D_{t+1} + P_{t+1} - P_t}{P_t} = \frac{D_{t+1} + P_{t+1}}{P_t} - 1.$$

Here,

$$\frac{D_{t+1}}{P_t} = \text{Income Yield}, \quad \frac{P_{t+1} - P_t}{P_t} = \text{Capital Gain}.$$

Gross Return

$$1 + R_{t \rightarrow t+1} = \frac{D_{t+1} + P_{t+1}}{P_t}.$$

Expected Return and Standard Deviation

$$\mathbb{E}(r) = \sum p(s) r(s), \quad \text{Var}(r) = \sigma^2 = \sum p(s) [r(s) - \mathbb{E}(r)]^2.$$

Under normality assumptions, standard deviation is a complete measure of risk for diversified portfolios.

Risk Premium

$$\text{Risk Premium} = \mathbb{E}[r] - R_f.$$

Excess Return

$$\text{Excess Return} = R_i - R_f.$$

Sharpe Ratio

$$\text{Sharpe Ratio} = \frac{\text{Risk Premium}}{\text{SD of excess return}} = \frac{\mathbb{E}[r] - R_f}{\sigma}.$$

It measures the risk-return trade-off for diversified portfolios (less suitable for single assets).

Example: Long-Term Bond

Face value = \$100, yearly coupon = \$2, bought at \$100 at beginning of Year 1. Suppose interest rate goes up after 1 year and bond price falls to \$98.5:

$$\text{Gross Return} = \frac{2 + 98.5}{100} = 1.005,$$

$$\text{Net Return} = 1.005 - 1 = 0.005 = 50\text{b.p.}$$

$$\text{Income Yield} = \frac{2}{100} = 2\%, \quad \text{Capital Gain} = \frac{98.5 - 100}{100} = -1.5\%.$$

Compounding Rate of Return

If the return each quarter is 8%, the one-year compound return is $(1.08)^4 = 1.3605$, i.e. an annualized 36%, not 32%.

Measuring Portfolio Return

$$R_{p,t \rightarrow t+1} = \sum_{n=1}^N w_n R_{n,t \rightarrow t+1},$$

where $\sum_{n=1}^N w_n = 1$.

2. Leverage

A negative portfolio weight ($w_n < 0$) on some asset indicates leverage or short-selling.

Money Market Asset

Borrowing at the risk-free rate R_f implies $w_f < 0$ is a leveraged position.

Buying on Margin

Suppose there is a risky asset (i) and a risk-free asset (f).

- *Pure long position example:* $w_i = 0.6$, $w_f = 0.4$.
- *Leveraged position example:* $w_i = 1.6$, $w_f = -0.6$.

Margin:

$$m = \frac{\text{Equity invested}}{\text{Value of Asset}} = \frac{\text{Total contributed by investor}}{\text{Total value of investment}} = \frac{1}{1.6} = 62.5\%.$$

$$\text{Leverage} = \frac{1}{m}.$$

Features: Stocks can be bought on margin from a broker at rate R_f (plus fees). If the asset price falls too much, the broker makes a margin call or forces liquidation.

Return on Leveraged Purchases

If we put up fraction m of the investment in the risky asset and borrow the rest at rate R_f , the gross return (per \$1) is:

$$\frac{1}{m}(1 + R_i) - \left(\frac{1}{m} - 1\right)(1 + R_f) = \frac{R_i - (1 - m)R_f + m}{m}.$$

Hence the *net return* is

$$\frac{R_i - (1 - m)R_f}{m} = R_f + \frac{1}{m}(R_i - R_f).$$
$$\mathbb{E}\left(R_f + \frac{1}{m}(R_i - R_f)\right) = R_f + \frac{1}{m}\mathbb{E}[(R_i - R_f)].$$

The standard deviation is also amplified by $\frac{1}{m}$.

Margins in Practice

- Initial margins on stocks are typically quite high due to regulation. Hedge funds find ways around this.
- Real estate can have low margins (high leverage) because property is a good collateral.

Leverage Cycle (2008 GFC): Low margins (high leverage) before the crisis raised housing-asset prices. Then margin requirements rose during the crisis, depressing prices further.

3. Short-selling

Another way to have $w_i < 0$: profit from a future price drop.

Procedures

1. Borrow shares from an institution.
2. Sell the stock, receive cash.
3. Deposit 102% of the value of the stock as collateral in a margin account.
4. Pay any dividend of the stock to the original owner.
5. Later buy back the shares, return them, and reclaim the deposit (with interest at R_f).

If the stock price falls, one profits.

Return to Short-selling: Per-share Method

Let

S_0 = Initial stock price, S_T = Repurchase price at period T ,

R_f = Annual risk-free rate, m = Margin requirement, D = Dividend per share = $S_0 \times$ Dividend rate.

$$\text{Number of shares borrowed} = \frac{\text{Value of share borrowed}}{S_0},$$

$$\text{Deposit collateral per share} = m S_0 (1 + R_f).$$

ROI:

$$ROI = \frac{S_0 (1 + R_f) - (S_T + D)}{m S_0}.$$

Generalization to Multiple Period ROI:

$$ROI = \frac{S_0 (1 + R_f)^T - S_T - \sum_{t=1}^T D_t (1 + R_f)^{T-t}}{m S_0},$$

where $D_t = S_0 \times \text{Div. yield}_0 + S_1 \times \text{Div. yield}_1 + \dots + S_{T-1} \times \text{Div. yield}_{T-1}$ if dividends are paid annually.

Short-selling and Regulation

Short-selling is sometimes banned during downturns, but many argue it improves liquidity and helps reveal negative information early.

4. Proof: No-Arbitrage Condition

Suppose there are people willing to buy a portfolio W for a price

$$p_W > \sum_{n=1}^N w_n p_n.$$

One could buy each component of the portfolio individually at the sum of those component prices, then immediately sell the bundle for p_W , profiting with no risk. In equilibrium, no such mispricing persists, so

$$p_W = \sum_{n=1}^N w_n p_n \quad (\text{No-arbitrage portfolio price}).$$

5. Arbitrage in Practice

Realistically, a margin (collateral) is always required, so a true “zero-equity” arbitrage rarely exists if markets function well.

Arbitrage in Theory vs. Practice

In theory, the market is assumed not to allow any risk-free profit. If any appear, rational players exploit it until it disappears. In practice, costs such as broker fees, bid-ask spreads, or short-selling constraints can disrupt perfect no-arbitrage.

6. Failures of “No-Arbitrage” in the Real World

- **U.S. Municipal Bonds vs. Corporate Bonds:** Tax-exempt interest, legal restrictions on shorting municipal debt, and other regulations create apparent yield discrepancies.
- **Mutual Funds:** Certain mutual fund shares can trade at premiums to their NAV.
- **Royal Dutch and Shell:** Dual-listed shares paying the same dividend sometimes traded at different prices, a perceived violation of LOOP. Short-selling difficulty (on one share) or index-inclusion effects can be partial explanations.

ECON0048 Lecture 3 Revision Note

Ambrose W

1 The Bond Market

1.1 Definition

- A market for long-term debt securities.
- Typically involves various currencies and maturities.
- Liquidity can range from very high to extremely low.
- Commonly issued in \$1,000 denominations or higher, so it is possible for individuals to hold them directly.
- Bonds exhibit a wide range of credit quality, from AAA-rated to junk.
- Typically traded over-the-counter (OTC) through dealers.
- Standard bonds often pay fixed coupons (interest).

2 Government Bonds

2.1 Basic Features

- Promise to repay principal at maturity.
- Typical components:
 - Annual or semi-annual coupon.
 - Face value = Principal + Coupon.
 - Coupon is fixed in nominal terms for standard bonds.
 - TIPS coupon is adjusted for inflation.

2.2 Markets for Government Bonds

- **Primary Market:** Newly issued bonds are sold at a bond auction. Only certain maturities are available.
- **Secondary Market:** All maturities are available; bonds may trade at discount or premium depending on market conditions.

2.3 Sovereign Risk Premium

- Defined as:

$$(\text{10-Yr Yield from country } i) - (\text{10-Yr Yield from U.S. or Germany})$$

2.4 Primary Market Outcomes

- The design of bond coupons in relation to market conditions influences the outcome of the bond auction.
- Price at issuance can equal principal (selling “at par”) or differ:
 - **Selling at discount:** Bond price < Face value (occurs if yield < risk-free rate).
 - **Selling at premium:** Bond price > Face value (occurs if yield > risk-free rate).

2.5 Secondary Market

As time passes, a bond’s remaining maturity changes, so newly issued bonds may have different yields. Consequently, older bonds can trade at discount or premium as market conditions and interest rates fluctuate.

2.6 Yield Curve

- **Standard Yield Curve:** Longer-term bonds generally offer higher annualized interest rates, reflecting compensation for longer holding periods.
- **Reversed Yield Curve:** Longer-term bonds offer lower annualized rates under certain expectations (e.g., if future short-term rates are expected to drop below current levels).

3 Corporate Bonds

- Firms with solid credit ratings are unlikely to default but risk always remains (e.g. Lehman Brothers).
- Firms with lower credit quality may default; higher promised yields compensate investors for this risk.
- Under default, bondholders take over the firm (or its remaining value).
- Interest-rate spreads can widen dramatically during crises, creating “flight to quality”.

4 Equities

4.1 Features

- Shares issued by corporations, representing ownership.
- Entitle holders to dividends and voting rights.
- Residual claims: only receive a payout after all other obligations (e.g. salaries, bond payments, taxes) are met.
- Limited liability: share price cannot go below zero, so there is no additional liability to shareholders.

4.2 Equity Issuance

- Initial Public Offering (IPO): First-time share issuance.
- Seasonal Equity Offering (SEO): Subsequent issuance of shares.
- Investment banks underwrite shares, set a target price, and distribute them to investors. They also manage the investor base to reduce the likelihood of mass dumping.

5 Indexes

5.1 Portfolio Return Measurement

- For a portfolio p , its return from t to $t + 1$ is:

$$R_{p,t+1} = \sum_{n=1}^N w_{n,t} R_{n,t+1}$$

- Here,

$R_{n,t+1}$ = Return of asset n at period $t+1$, $w_{n,t}$ = Portfolio weight of asset n at period t .

5.2 Index Features

An index is characterized by its set of constituent assets and their weighting scheme.

5.3 Weighting Schemes

5.3.1 Price-Weighted

$$w_n = \frac{P_n}{p_1 + \cdots + p_N}$$

- Example: DJIA.
- This weighting requires frequent rebalancing because the number of shares held must adjust continuously to maintain the ratio.

5.3.2 Value-Weighted

$$w_n = \frac{V_n}{V_1 + \dots + V_n}$$

where V_n denotes the market value (price per share times outstanding shares). Examples include S&P 500, FTSE 250, SSE, and HSI. Value-weighted portfolios do not require rebalancing.

6 Market Completeness

6.1 Definition of a Complete Market

A market is complete if any asset payoff can be replicated by a linear combination of traded securities (“primitive state-contingent payoffs”), which requires full column rank among securities.

6.2 Are Markets Complete?

- In principle, there must be at least as many non-redundant assets as possible states of the world.
- However, in practice, there can be countless states and fewer tradable assets.
- Irrelevant dimensions of uncertainty may be ignored if they do not affect payoffs.

6.3 Coin Toss Example

- A twice-repeated coin toss produces four possible outcomes.
- If payoff depends only on the *total number of heads*, states can be reduced; in that case, fewer securities might be necessary.

6.3.1 Dynamic Trading

- Even if there are more states than assets at the final date, one can trade dynamically across periods to achieve completeness under certain conditions.
- Example with two Arrow–Debreu (AD) securities (“Head” AD-H and “Tail” AD-T). Specific trading strategies can replicate payoffs of more final states than the number of securities implies.

6.4 Conclusion on Market Completeness

- Dynamic trading in a small set of assets (bonds, equities) can effectively complete the market under a binomial or trinomial model.
- Real-world frictions like short-selling costs, verification challenges, and complexity of certain securities (e.g. securitized debt, MBS, CDS) often render markets incomplete.

6.5 Implications of an Incomplete Market

- Not all payoffs can be replicated.
- Full column rank or sufficient liquidity is often violated.
- Despite these limitations, complete markets remain a standard theoretical tool for valuing complex securities.

7 From Arbitrage to Equilibrium Prices

- The price of a portfolio is

$$p^W = \sum_{n=1}^N w_n p_n$$

- Actual asset prices p_n ultimately depend on supply, demand, and uncertainty over future returns.

8 Modeling Preferences for Risk

8.1 Assumptions

- Single consumption good.
- One period with initial consumption W_0 .
- After returns are realized, consumption = $W_0 + (\text{Investment returns})$.
- Preferences captured by expected utility of wealth.

8.2 Preferences Using Expected Utility

- A twice-differentiable, increasing utility function $U(W)$ is assumed.
- Investors maximize $\mathbb{E}[U(W)]$.
- In the two-state scenario ($s = 1, 2$):

$$\mathbb{E}[U(W)] = \pi_1 U(W_1) + \pi_2 U(W_2).$$

8.3 Risk Aversion Criteria

Consider a zero-mean gamble $\pm h$ with probability $1/2$. A risk-averse individual rejects this zero-mean gamble:

$$U(W_0) > \mathbb{E}[U(W_0 + G)] = \frac{1}{2}U(W_0 + h) + \frac{1}{2}U(W_0 - h).$$

The implication:

$$U(W_0) - U(W_0 - h) > U(W_0 + h) - U(W_0).$$

Hence, losing wealth reduces utility more than gaining the same amount increases it.

8.4 Risk Neutrality

- A risk-neutral individual is indifferent to the gamble.
- The utility function is linear, so the expected utility equals the utility of the expected value:

$$U(W_0) = \mathbb{E}[U(W_0 + G)].$$

- Risk neutral investor is also indifferent between two options with the same expected profit.

8.5 Risk Aversion and Utility Functions

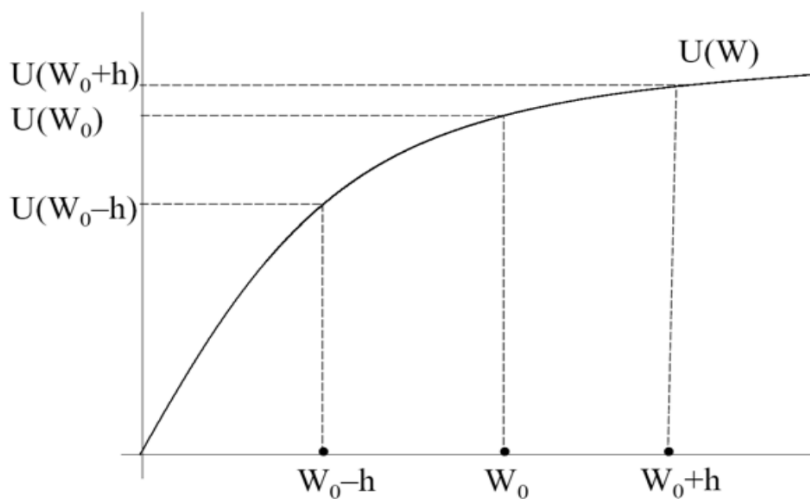


Figure 1: Risk averse utility function

- Risk-averse: $U(W)$ is concave and $U''(W) < 0$.
- Risk-neutral: $U(W)$ is linear.
- Risk-loving: $U''(W) > 0$.

8.6 Measuring Risk Aversion

- More negative $U''(W)$ indicates higher aversion to risk.

8.6.1 Absolute Risk Aversion (Arrow–Pratt)

$$A(W) = -\frac{U''(W)}{U'(W)}.$$

8.6.2 Relative Risk Aversion

$$R(W) = W \cdot A(W).$$

8.6.3 Risk Tolerance

$$\text{Risk tolerance} = \frac{1}{A(W)}.$$

8.7 Examples of Utility Functions

- **Linear:** $U(W) = a + bW$. Implies risk neutrality ($U''(W) = 0$).
- **Quadratic:** Not valid for general discussions due to turning points.
- **Exponential:** $U(W) = -\exp(-aW)$, which has constant absolute risk aversion $A(W) = a$.
- **Power (CRRA):**

$$U(W) = \frac{W^{1-\gamma} - 1}{1-\gamma},$$

constant relative risk aversion,

$$A(W) = \frac{\gamma}{W} \implies R(W) = \gamma.$$

A special case is $\ln(W)$, which has $R(W) = 1$.

9 Summary of Key Points

- Corporate bonds have priority in payoffs; equity is residual.
- Bond payoffs are promised, whereas equity payoffs are not fixed.
- As uncertainty is resolved, dynamic trading can sometimes achieve a complete market, even if full rank is not satisfied at the outset.

ECON0048 Lecture 4 Notes

Ambrose W

Lecture 4.1: Probability and Sample Means

Probability

- **Frequency:** The fraction of times an event occurs in total.
- **Probability:** The frequency that would occur if the same experiment is repeated infinitely many times.
- As the number of trials $\rightarrow \infty$, frequency \rightarrow probability.

Arithmetic Average Return

- Define the sample mean of the return. Suppose we have historical observations for periods $t = 1, \dots, T$ and observe the return R_t in each period. The arithmetic mean is:

$$\bar{R} = \frac{1}{T} \sum_{t=1}^T R_t, \quad 1 + \bar{R} = \frac{1}{T} \sum_{t=1}^T (1 + R_t).$$

Geometric Average Return

- Annualized cumulative return:

$$(1 + \bar{G})^T = \prod_{t=1}^T (1 + R_t).$$

- Hence, the annualized net geometric return is:

$$\bar{G} = \left(\prod_{t=1}^T (1 + R_t) \right)^{\frac{1}{T}} - 1.$$

Arithmetic vs. Geometric: Which is Bigger

- By taking natural logarithms of the geometric average,

$$\ln(1 + \bar{G}) = \frac{1}{T} \sum_{t=1}^T \ln(1 + R_t).$$

- Using Jensen's Inequality (since $\ln(\cdot)$ is concave):

$$\ln\left(\frac{1}{T} \sum_{t=1}^T X_t\right) \geq \frac{1}{T} \sum_{t=1}^T \ln(X_t), \quad \text{where } X_t = 1 + R_t.$$

- It follows:

$$\bar{R} \geq \bar{G}.$$

Difference Between \bar{R} and \bar{G} : Rule of Thumb

- For a log-normal distribution:

$$\bar{G} \approx \bar{R} - \frac{1}{2} \text{var}_{\text{sample}}(R).$$

- The difference grows with greater volatility or longer holding period.

Log Gross Returns

- Define **log gross returns** as $\ln(1 + R_t)$.
- Returns over longer horizons become additive in logs:

$$\ln((1 + R_s)(1 + R_u)) = \ln(1 + R_s) + \ln(1 + R_u).$$

- For most assets, log gross returns are approximately normal over a month or more.

Lognormality and the Median

- If $\ln(1 + R_t) \sim \mathcal{N}(\mu, \sigma^2)$, then $1 + R_t$ is lognormal.
- By Central Limit Theorem, the product of many independent gross returns (i.e. the sum of their logs) tends to normal.

Law of Large Number & Probability of Return Greater than Geometric Mean

- As $t \rightarrow \infty$,

$$\ln(1 + R_t)_{\text{sample mean}} = \frac{1}{T} \sum_{t=1}^T \ln(1 + R_t) \longrightarrow \mathbb{E}[\ln(1 + R_t)].$$

- By a large-sample argument,

$$\Pr(\ln(1 + R_t) > \ln(1 + \bar{G})) = \frac{1}{2} \quad \text{for large } T.$$

- Hence, about half the time R_t is above \bar{G} and half the time below, if we look at many periods.

Distribution of R_t

- Typically right-skewed (asymmetric).
- $\Pr(R_t > \bar{R}) < \frac{1}{2}$ for right-skewed distributions.

Which Average is Better

- For measuring past performance over multiple periods, \bar{G} (the geometric average) is more relevant because it depends on the cumulative return.
 - \bar{R} is the expected return if holding the asset for a single randomly selected period.
-

Lecture 4.2

A Statistical View of Financial History

- Arithmetic mean of (Real) Returns on Different Assets (1926–2006):

$$\text{T-bill} = 0.8\%, \quad \text{10-Yr Treasury Bond} = 2.9\%, \quad \text{Stocks} = 8.8\%.$$

- Trends:
 - Volatility: Stock > Long-term bonds > Short-term bonds.
 - Return: Stock > Long-term bonds > Short-term bonds.
- Gold:
 - Volatility roughly the same as stocks ($\sim 19\%$ per annum),
 - But mean return is much lower than stocks ($\sim 3.8\%$).
- To analyze the discrepancy in gold returns, consider the variance (risks) at the portfolio level and correlation with other assets.

Optimality and Equilibrium: Simplest Portfolio Problem (2-State Model)

- Consider a 2-state scenario with 2 Arrow-Debreu (AD) securities (i.e. Complete market).
- Assume a concave utility U with risk aversion.
- Investor with initial wealth W_0 buys W_1 units of AD1 and W_2 units of AD2 at state prices q_1 and q_2 .

- Objective:

$$\max_{W_1, W_2} \mathbb{E}[U(W)] = \pi_1 U(W_1) + \pi_2 U(W_2),$$

subject to the budget constraint

$$W_0 = W_1 q_1 + W_2 q_2.$$

- First-order condition:

$$\pi_1 U'(W_1) + \pi_2 U'(W_2) \left(-\frac{q_1}{q_2}\right) = 0,$$

$$\frac{U'(W_1)}{U'(W_2)} = \frac{q_1/\pi_1}{q_2/\pi_2}.$$

- Interpretation: LHS is the marginal utility ratio, RHS is the price/probability ratio across states.

Lucas's Thought Experiment & Gold

- Suppose W_1 and W_2 are exogenously fixed.
- Investors cannot freely move resources between states, but they try to do so by trading AD securities, pushing up or down state prices until indifference.
- Equilibrium: Asset prices depend on probabilities and marginal utilities:

$$\frac{q_1}{q_2} = \frac{\pi_1}{\pi_2} \times \frac{U'(W_1)}{U'(W_2)}.$$

- Probability of a state $\uparrow \implies$ state price \uparrow .
- Marginal utility $\uparrow \implies$ state price \uparrow .
- Since GLD provides insurance against unpleasant states (*the states where the economy is not doing well*), so it has a lower expected return despite similar volatility to stocks.

Equilibrium with Risk-Neutral Investors

- When investors are risk-neutral, marginal utility is constant. Then

$$\frac{q_1}{q_2} = \frac{\pi_1}{\pi_2}.$$

- State prices equal probabilities (up to a scale factor).

Equilibrium with Risk-Averse Investors

- $$\frac{q_1}{q_2} = \frac{\pi_1}{\pi_2} \times \frac{U'(W_1)}{U'(W_2)}.$$
- More probability for a state \implies higher price. Unpleasant state \implies higher marginal utility \implies higher price.

State Prices to Asset Prices

- Relationship:

$$p_n = q_1 X_{n1} + q_2 X_{n2}.$$

- Conclusion: An asset that pays in unpleasant states is more valuable (provides insurance) and thus has a lower expected return. An asset that pays in good states is less valuable and must offer a higher expected return.

The Stochastic Discount Factor (SDF)

- Rewrite asset price vs. state price:

$$p_n = \pi_1 \frac{q_1}{\pi_1} X_{n1} + \pi_2 \frac{q_2}{\pi_2} X_{n2} = \pi_1 M_1 X_{n1} + \pi_2 M_2 X_{n2} = \mathbb{E}[M X_n].$$

- $M_s = \frac{q_s}{\pi_s}$ is the SDF in state s .
- If risk-neutral, $M_s = 1$. If risk-averse, $M_s \neq 1$.
- M_s is proportional to marginal utility in that state.
- Price of any asset is the expected product of SDF and payoff.

Defining the Insurance Premium (Additive Risk)

- Let Z be a random variable with $\mathbb{E}[Z] = 0$ and $\text{Var}(Z) = \sigma^2$. The expected utility of facing Z is $\mathbb{E}[U(W_0 + Z)]$.
- An *additive* insurance premium π is defined by the indifference condition:

$$U(W_0 - \pi) = \mathbb{E}[U(W_0 + Z)].$$

- Scaling risk: if the risk is εZ , define $\pi(\varepsilon)$ by the following indifference condition:

$$U(W_0 - \pi(\varepsilon)) = \mathbb{E}[U(W_0 + \varepsilon Z)].$$

- Use Taylor expansions around small ε . For second-order approximation:

$$\pi(\varepsilon) \approx \frac{1}{2} A(W_0) \text{Var}(\varepsilon Z),$$

where $A(W_0)$ is absolute risk aversion.

- Under risk neutrality ($A(W_0) = 0$), $\pi(\varepsilon) = 0$.

Relative Risk Aversion and Multiplicative Risk

Suppose random variable $Z = zW_0$, with $\mathbb{E}[Z] = 0$. Total wealth when carry risk = $W_0(1+z)$

For example:

$$z = \begin{cases} 0.1, p = 0.5 \\ -0.1, p = 0.5 \end{cases} \in Z$$

Taking the indifference condition:

$$U(W_0 - \pi(\epsilon)) = \mathbb{E}[U(W_0 + \epsilon z W_0)]$$

By applying similar Taylor approximation technique, we obtained $\pi(\epsilon)$

$$\pi(\epsilon) \simeq \frac{1}{2} \text{Var}(\epsilon z W_0) A(W_0) = \frac{1}{2} \text{Var}(\epsilon z) W_0^2 A(W_0)$$

Rearranging the last equation, we have

$$\frac{\pi(\epsilon)}{W_0} \simeq \frac{1}{2} \text{Var}(\epsilon z) R(W_0)$$

Therefore, **relative risk aversion $R(W_0)$ measures the percentage of your wealth you are willing to give up in order to avoid a multiplicative risk.**

Portfolio Choice with Expected Utility

- For a set of portfolio weights $\{w_n\}$, the portfolio return is R_w . The investor solves:

$$\max_{R_w \in C} \mathbb{E}[U(W_0(1 + R_w))].$$

- By a second-order Taylor approximation (assuming R_w^2 is small),

$$U(W_0(1 + R_w)) \approx U(W_0) + U'(W_0) W_0 R_w + \frac{1}{2} U''(W_0) W_0^2 R_w^2.$$

- Taking expectations and dropping constants/scales, we get the well-known *mean-variance* form:

$$\mathbb{E}(R_w) - \frac{1}{2} \text{RRA} \sigma_{R_w}^2,$$

where RRA is the relative risk aversion and $\sigma_{R_w}^2$ is the variance of the portfolio return.

Conclusion

- If an asset pays off in states where marginal utility is high, it is more valuable. Investors' desire to allocate wealth to those high-MU states raises the asset price, lowering its expected return.
- This explains why gold can have a lower expected return (it pays when times are bad), while stocks pay well in good states and must offer a higher expected return.

- The price of any asset can also be understood via the SDF approach: it is the expectation of the product between the payoff and a random variable (the SDF) proportional to marginal utility.
- Coefficients of absolute and relative risk aversion indicate how much one is willing to pay to avoid risk, either in dollar terms (absolute) or as a fraction of wealth (relative).

Lecture 5.1: Risk & Diversification

Investment Horizon

Question: How does the measuring interval (investment horizon) matter? For example, does an investor with a one-year horizon choose a different portfolio weight than an investor with a one-quarter horizon?

Key Idea: Serial Independent Returns. If returns are i.i.d. (independent and identically distributed; uncorrelated over time), the mean and variance of log returns both grow in proportion to the investment horizon, implying that the horizon does not affect the optimal choice of weights in a mean-variance framework.

$$\text{Total return over } H : \sum_{t=1}^H \log(1 + R_t).$$

Expected total return:

$$\mathbb{E} \left[\sum_{t=1}^H \log(1 + R_t) \right] \approx H \cdot \mathbb{E}[\log(1 + R_t)] \approx H \cdot \mathbb{E}[R_t].$$

Variance of total return:

$$\text{Var} \left(\sum_{t=1}^H \log(1 + R_t) \right) \approx H \cdot \text{Var}(\log(1 + R_t)) \approx H \cdot \text{Var}(R_t).$$

Optimal Portfolio Weight with Horizon H :

$$w_{p,H} = \frac{H \cdot \mathbb{E}[R_p] - H \cdot R_f}{H \cdot \sigma_p^2 \times RRA} = \frac{\mathbb{E}[R_p] - R_f}{\sigma_p^2 \times RRA}.$$

Thus, the optimal weight does not actually change with H when returns are time-uncorrelated.

Sharpe Ratio over Different Horizons.

$$S_{p,H} = \frac{H \cdot (\mathbb{E}[R_p] - R_f)}{\sqrt{H} \sigma_p} = \sqrt{H} (S_p).$$

The Sharpe ratio grows with \sqrt{H} . Yet, even though the Sharpe ratio increases over a longer horizon, the *optimal portfolio weight remains unchanged* if returns are i.i.d.

Diversification

Idiosyncratic (Specific/ Diversifiable) Risk. Risk that affects only a single company, asset, or industry, due to firm-specific factors. It is diversifiable through holding multiple assets.

Systemic (Non-diversifiable) Risk. Potential for collapse or severe disruption of the financial system or economy as a whole, driven by broad macroeconomic factors. It is non-diversifiable.

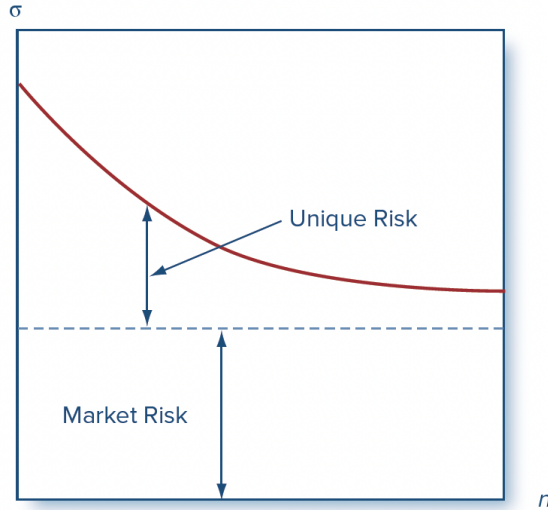


Figure 1: Capital Allocation Line

Simple Illustration

Consider two risky stocks with the same expected return R :

$$R_1 = R + \varepsilon_1, \quad R_2 = R + \varepsilon_2,$$

where ε_1 and ε_2 are zero-mean, uncorrelated random variables with the same variance σ^2 .

- **Non-diversified portfolio:** Putting everything in one stock yields variance σ^2 .
- **Equal-weight portfolio:** Weight each stock by $\frac{1}{2}$, so portfolio return is $\frac{R_1+R_2}{2}$. Its variance becomes $\frac{1}{2}\sigma^2$, showing that diversification halves the variance in this simple case.

For N such uncorrelated stocks, equally weighted,

$$\text{Var}\left(\frac{1}{N} \sum_{i=1}^N R_i\right) = \frac{\sigma^2}{N}.$$

Hence variance can be made quite small with large N if risks are uncorrelated and purely idiosyncratic.

Systematic Shock

$$R_i = R + \varepsilon_{\text{sys}} + \varepsilon_i,$$

where ε_{sys} is a common systematic component. Diversification cannot eliminate this part; as $N \rightarrow \infty$, the portfolio variance approaches $\text{Var}(\varepsilon_{\text{sys}})$. Thus, one can eliminate idiosyncratic risk through diversification, but not systematic risk.

Correlation and Diversification

If each stock has variance σ^2 and correlation ρ , then systematic variance is $\rho \sigma^2$ and idiosyncratic variance is $(1 - \rho) \sigma^2$. The benefit of diversification exists whether correlation is positive, zero, or negative. The lower the correlation, the greater the potential reduction in overall portfolio variance.

Excess variance ratio for an equally weighted portfolio of N stocks:

$$\% \text{Excess Variance} = \frac{(1 - \rho) \sigma^2}{N \rho \sigma^2} = \frac{1 - \rho}{N \rho}.$$

As correlation ρ decreases or N increases, the excess variance declines.

Changing Correlation

Historical evidence suggests that average correlation among individual stock returns declined from the 1960s to mid-1990s, implying more stocks needed for adequate diversification. Since the mid-2000s, some of that decline has reversed, partially increasing correlations again.

Conclusion of Lecture 5.1

- **Investment Horizon:** Under i.i.d. returns, the optimal weight in risky assets does not change with horizon.
- **Diversification:** Diversifies away idiosyncratic risk, but does not eliminate systemic risk.

1. Mean-Variance Analysis with One Risky and One Risk-Free Asset

Setup: Assume there is a risk-free asset with return R_f and a single risky asset with return R_p .

1.1 Portfolio Return

A portfolio consisting of weight w_p in the risky asset and $(1 - w_p)$ in the risk-free asset has return:

$$R_w = w_p R_p + (1 - w_p) R_f = R_f + w_p (R_p - R_f).$$

Short-selling is assumed to be costless and permitted, so w_p can be less than 0 or greater than 1.

1.2 Mean of the Portfolio

The expectation of the portfolio return is:

$$\mathbb{E}[R_w] = R_f + w_p (\mathbb{E}[R_p] - R_f).$$

The quantity $\mathbb{E}[R_w] - R_f$ is referred to as the portfolio's risk premium. Hence,

$$\mathbb{E}[R_w] - R_f = w_p (\mathbb{E}[R_p] - R_f).$$

1.3 Variance of the Portfolio

Because the risk-free asset has zero variance, the variance of the portfolio return is driven solely by the risky part:

$$\sigma_w^2 = \sigma_p^2 w_p^2.$$

1.4 Choosing the Portfolio Weight

An investor seeks to maximize utility (or equivalently, an approximation to utility that accounts for mean and variance). From Lecture 4, the approximate investor's preference is often written as:

$$U = \mathbb{E}[R_w] - \frac{1}{2} \text{RRA} \sigma_{R_w}^2.$$

Capital Allocation Line (CAL): By combining the portfolio mean and variance expressions, we have a linear relationship between the expected return and the standard deviation of the portfolio:

$$\mathbb{E}[R_w] = R_f + \frac{\mathbb{E}[R_p] - R_f}{\sigma_p} \sigma_w.$$

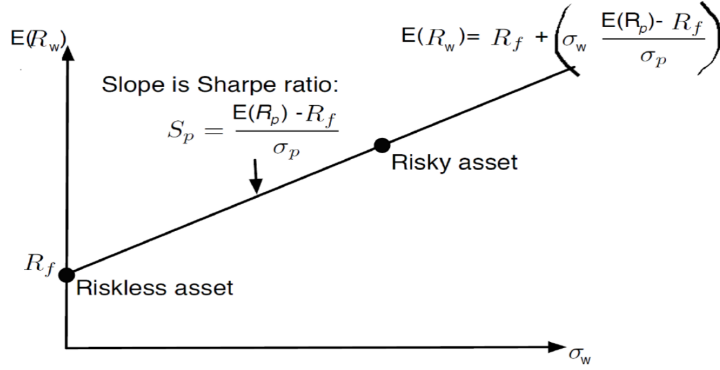


Figure 2: Capital Allocation Line

Sharpe Ratio S_p

The slope of the CAL is the Sharpe Ratio:

$$S_p = \frac{\mathbb{E}[R_p] - R_f}{\sigma_p},$$

also known as the reward-to-volatility ratio. In the one-risky-one-risk-free-asset case, an optimal portfolio will inherit the Sharpe ratio of the underlying risky asset.

Solution of the Optimization

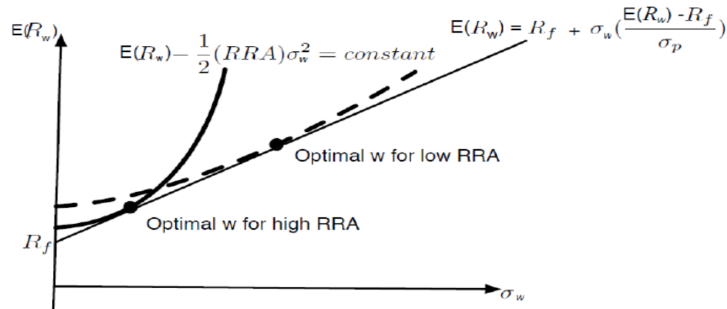


Figure 3: Capital Allocation Line

All investors who share the same risky asset and risk-free asset face the same Sharpe ratio. Differences in risk aversion lead to different optimal weights but not to different Sharpe ratios.

Analytical Solution: Substituting the mean and variance into the approximate preference, the first-order condition yields:

$$\mathbb{E}[R_p] - R_f - \text{RRA} w_p \sigma_p^2 = 0.$$

Hence,

$$w_p = \frac{\mathbb{E}[R_p] - R_f}{\text{RRA} \sigma_p^2} = \frac{S_p}{\text{RRA} \cdot \sigma_p}.$$

Implications and Summary

If $\mathbb{E}[R_p] - R_f > 0$, it is always optimal to hold some of the risky asset. If $w_p > 1$, it implies leveraging (buying on margin).

The **optimal portfolio risk** (the standard deviation of the chosen portfolio) is:

$$\sigma_w = w_p \sigma_p = \frac{S_p}{\text{RRA}}.$$

This shows how the optimal risk taken is proportional to the Sharpe ratio and inversely proportional to risk aversion (RRA).

In practice, $1 < \text{RRA} < 100$ is often a reasonable range, implying typical portfolio standard deviations between 10% and 100% of the available Sharpe ratio.

2. Mean-Variance Analysis with Two Risky Assets (Without a Risk-Free Asset)

Consider two risky assets with returns R_1 and R_2 . Their expected returns are $\mathbb{E}[R_1]$ and $\mathbb{E}[R_2]$, and their (Pearson) correlation coefficient is

$$\rho_{12} = \frac{\text{Cov}(R_1, R_2)}{\sigma_1 \sigma_2}.$$

2.1 Portfolio Return

With weights w_1 and w_2 in assets 1 and 2 respectively (where $w_1 + w_2 = 1$), the portfolio return is:

$$R_p = w_1 R_1 + w_2 R_2.$$

Mean of the Portfolio

By linearity of expectation:

$$\mathbb{E}[R_p] = w_1 \mathbb{E}[R_1] + w_2 \mathbb{E}[R_2],$$

Variance of the Portfolio

$$\sigma_p^2 = w_1^2 \text{Var}(R_1) + w_2^2 \text{Var}(R_2) + 2 w_1 w_2 \text{Cov}(R_1, R_2).$$

Equivalently:

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_1 \sigma_2 \rho_{12}.$$

Power of Diversification

If $0 < w_1, w_2 < 1$ and $\rho_{12} \leq 1$, one can show that σ_p^2 is always less than or equal to $(w_1\sigma_1 + w_2\sigma_2)^2$. Hence,

$$\sigma_p \leq w_1 \sigma_1 + w_2 \sigma_2,$$

with equality only if $\rho_{12} = 1$. If $\rho_{12} = -1$, the portfolio can potentially attain zero variance by a suitable choice of weights.

2.2 Global Minimum Variance Portfolio (GMV)

One can seek to minimize the variance:

$$\sigma_p^2 = w_1^2 \sigma_1^2 + (1 - w_1)^2 \sigma_2^2 + 2w_1(1 - w_1)\sigma_1\sigma_2\rho_{12}.$$

Solving $\frac{\partial \sigma_p^2}{\partial w_1} = 0$ gives a closed-form expression for the global minimum variance weights. In particular cases:

- If $\rho_{12} = 1$, one finds that

$$\sigma_p^2 = (w_1 \sigma_1 + (1 - w_1) \sigma_2)^2,$$

which can be driven to 0 when shorting is allowed and setting

$$w_{G1} = \frac{-\sigma_2}{\sigma_1 - \sigma_2} < 0, \quad w_{G2} = 1 - w_{G1} = \frac{\sigma_1}{\sigma_1 - \sigma_2}$$

- If $\rho_{12} = -1$, it is possible to achieve zero variance even with positive weights, if σ_1 and σ_2 perfectly offset each other in the portfolio.

$$w_{G1} = \frac{\sigma_2}{\sigma_1 + \sigma_2}$$

- **General case**

Suppose $\rho_{12} \in (-1, 1)$, then the portfolio std err cannot be lowered to naught.

Given that for a general ρ , σ_p^2 is a quadratic function of w_1 . We wish to find w_G that minimizes portfolio variance.

$$\sigma_p^2 = (w_1\sigma_1 + (1 - w_1)\sigma_2)^2 = w_1^2\sigma_1^2 + (1 - w_1)^2\sigma_2^2 + 2w_1(1 - w_1)\sigma_{12}$$

where $\sigma_{12} = \text{Cov}(R_1, R_2)$

$$w_{G1} = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}, \quad w_{G2} = 1 - w_{G1} = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

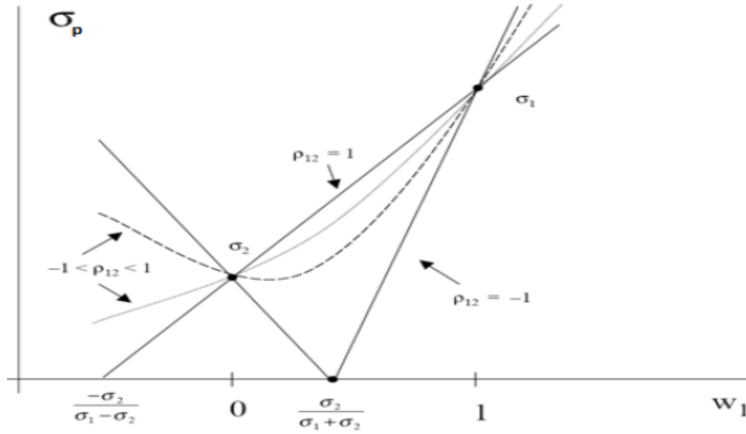


Figure 4: GMV Portfolio

Special Cases for GMV

- If $\text{Cov}(R_1, R_2) = 0$, then

$$w_{G1} = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \quad \text{and} \quad w_{G2} = 1 - w_{G1} = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}.$$

- If $\sigma_1^2 = \sigma_2^2$, then

$$w_{G1} = w_{G2} = \frac{1}{2}.$$

2.3 Efficient Frontier

Plotting $\mathbb{E}[R_p]$ against σ_p reveals a hyperbola-like shape. The **efficient frontier** is the “upper arm” of this hyperbola, the set of portfolios that gives the highest expected return for each given level of risk (standard deviation). Any point on the lower part is *dominated* by some point on the upper part that offers the same (or lower) risk but a higher expected return.

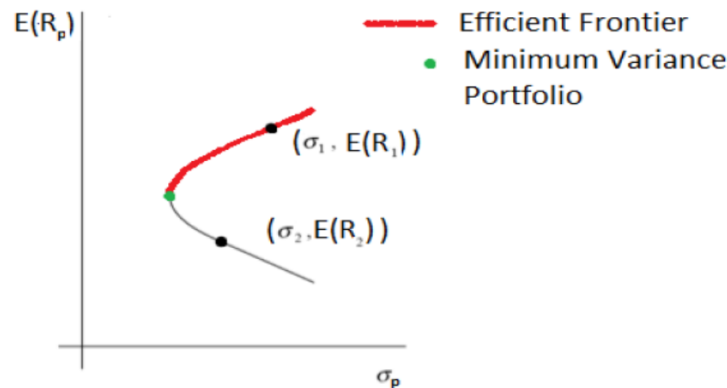


Figure 5: Efficient Frontier

2.4 Adding More Assets

Extending to three or more risky assets yields a higher-dimensional generalization, still forming a hyperbolic shape in mean-standard-deviation space. Adding assets not perfectly correlated with the existing set can only improve (or at least not worsen) the efficient frontier.

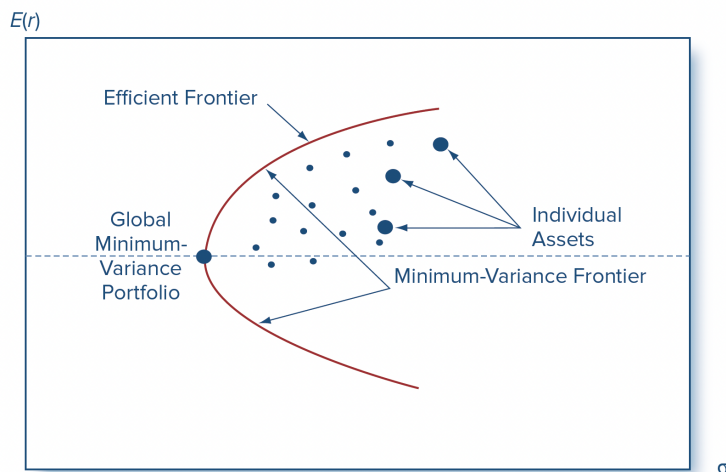


Figure 6: EF of many risky assets

3. Optimal Portfolio with One Risk-Free and Many Risky Assets

Now suppose there is a set of N risky assets plus a risk-free asset. We want to find a *complete portfolio* that maximizes investor utility. The fundamental idea is:

- First, find the **tangency portfolio** of the risky assets (the portfolio among those assets that yields the highest Sharpe ratio).
- Second, mix that tangency portfolio with the risk-free asset according to the investor's risk aversion.

3.1 Step-1: Tangency Portfolio

The tangency portfolio p solves:

$$\max_{w_1, \dots, w_N} S_p = \frac{\mathbb{E}[R_p] - R_f}{\sigma_p} \quad \text{subject to} \quad w_1 + \dots + w_N = 1.$$

Here, R_p is the return of the portfolio composed of the risky assets. In practice, one needs:

- The expected returns $\{\mathbb{E}[R_i]\}_{i=1}^N$,
- The variance-covariance matrix of those N assets.

For 2 Risky asset + One risk-free asset, the tangency portfolio is:

$$w_{p1} = \frac{[\mathbb{E}[R_1] - R_f] \sigma_2^2 - [\mathbb{E}[R_2] - R_f] \sigma_{12}}{[\mathbb{E}[R_1] - R_f] (\sigma_2^2 - \sigma_{12}) + [\mathbb{E}[R_2] - R_f] (\sigma_1^2 - \sigma_{12})}$$

3.2 Step-2: Optimal Complete Portfolio

After solving for $w_{p,i}$ in the tangency portfolio, one treats that tangency portfolio p as if it were a single risky asset. Then, using the same procedure as in Section 1, the investor combines p and R_f to find the final share w_p that maximizes:

$$\mathbb{E}[R_w] - \frac{1}{2} \text{RRA} \sigma_w^2.$$

All investors who face the same set of risky assets and the same risk-free asset (and who have the same beliefs about expected returns/covariances) choose *the same tangency portfolio* of risky assets. They then differ only in how much of that tangency portfolio vs. risk-free asset they hold.

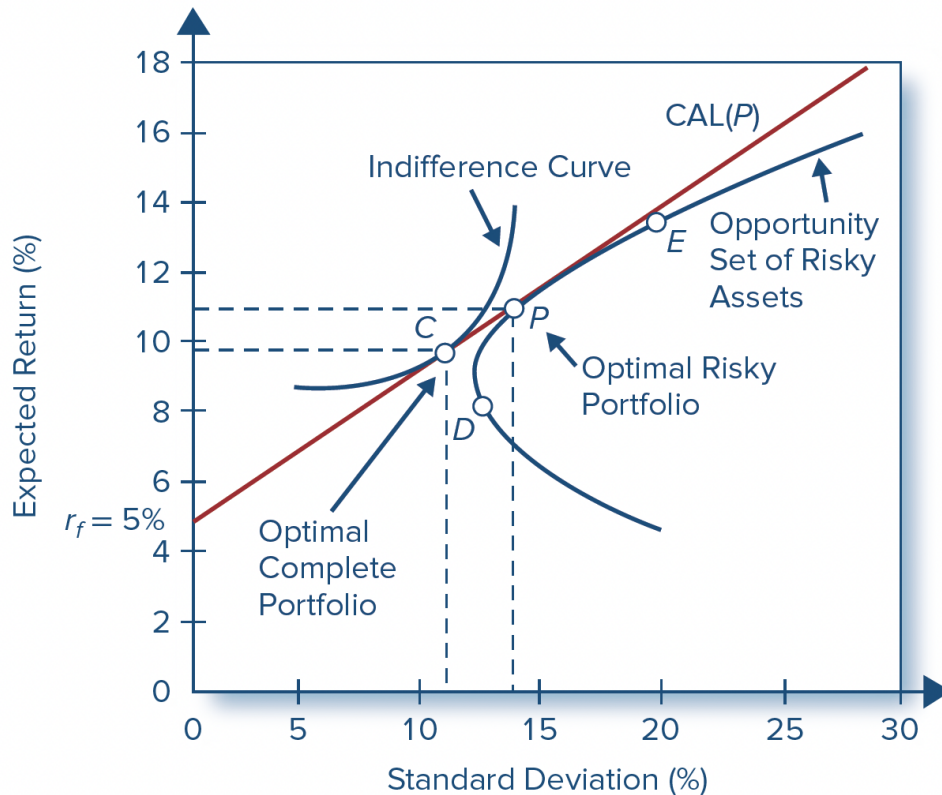


Figure 7: Optimal mean-variance complete portfolio choice

ECON0048: Mutual Fund Theorem and CAPM

Ambrose W

1 Lecture 6.1: Mutual Fund Theorem

1.1 Statement of the Mutual Fund Theorem

Mutual Fund Theorem (aka Separation Theorem): Once a risk-free asset is available, all investors who want to hold risky assets will hold the *same* proportions of those risky assets, or the *tangential portfolio*. The only difference among investors is in how much they invest in that tangential portfolio versus in the risk-free asset.

- If an investor chooses a fraction w_f in the risk-free asset, then the fraction $1 - w_f$ is invested in the common risky portfolio.
- The specific *weights* on different risky assets do not depend on an investor's risk aversion. Instead, *only* how much is placed in the overall risky portfolio (versus the risk-free asset) depends on risk aversion.

Implication: Conservative investors simply hold more of the risk-free asset and scale down their exposure to the common risky portfolio, but the *composition* of the risky portfolio itself is the same across all investors.

1.2 Proof of the Mutual Fund Theorem

Recall that we consider a given *Relative Risk Aversion* (RRA), and aim to solve:

$$\max_{w_1, w_2, \dots, w_n} \left(\mathbb{E}[R_p] - \frac{1}{2} RRA \cdot \sigma_p^2 \right)$$

subject to

$$w_f + \sum_{i=1}^n w_i = 1.$$

Here, w_f is the weight on the risk-free asset with return R_f , and w_i are weights on the risky assets with returns R_i .

Expected Return and the Constraint

$$\mathbb{E}[R_p] = w_f R_f + \sum_{i=1}^n w_i \mathbb{E}[R_i].$$

We use $w_f + \sum w_i = 1$ to eliminate w_f in the objective.

First-Order Condition (FOC)

Without proof, this maximization problem gives the *beta representation* for the portfolio w :

$$\frac{\mathbb{E}[R_i] - R_f}{\mathbb{E}[R_p] - R_f} = \frac{\text{Cov}(R_p, R_i)}{\sigma_p^2} \equiv \beta_i^w.$$

Conclusion

The weights w_i that solve this problem *do not depend on the investor's RRA*. Thus all investors choose the same *risky* portfolio weights. Their only difference is how much w_f (risk-free) they hold.

1.3 Beta Definition

Beta, denoted β_i^w , measures the covariance of asset i with the portfolio w , normalized by the portfolio variance:

$$\beta_i^w = \frac{\text{Cov}(R_w, R_i)}{\sigma_w^2}.$$

It quantifies *systematic* risk: the sensitivity of R_i 's excess return to R_w 's excess return.

1.4 Key Takeaway of Lecture 6.1

- All investors hold the same composition of risky assets (the *tangency portfolio*).
- Individual risk tolerance only affects the fraction of wealth allocated to the risk-free asset vs. the common risky portfolio, not the *mix* of the risky assets themselves.

2 Beta Representation Theorem

Statement of the Theorem

Suppose there is a risk-free asset f . Let w denote an optimal portfolio for an individual (with some risk-aversion). Then for any risky asset i :

$$\mathbb{E}[R_i] - R_f = \beta_{iw}(\mathbb{E}[R_w] - R_f)$$

where

$$\beta_{iw} \equiv \frac{\text{Cov}(R_w, R_i)}{\sigma_w^2}.$$

Key points:

- The risk premium of each asset (LHS) equals its “beta” with an optimal portfolio (β_{iw}) multiplied by the risk premium of that optimal portfolio.

- If an asset has positive covariance with the market (e.g. a stock), it provides a positive risk premium. Conversely, if it has negative covariance (e.g. insurance), its expected return is lower (or one effectively pays the risk premium).
- The theorem applies to any efficient portfolio w , including the tangency portfolio p of purely risky assets.
- The mean of asset i is thus “explained” by its covariance with the tangency portfolio.

3 Problems of Mean-Variance Optimization

Short-run Impreciseness

- In practice, estimates of the means (expected returns) over short samples can be very imprecise (“garbage in, garbage out”).
- Moments such as mean, variance, and covariances can change over longer periods, violating stationarity assumptions.

Huge Number of Parameters

Even if the moments are stable, the mean-variance optimization procedure for N assets requires:

- Estimating N expected returns (the first moments).
- Estimating $\frac{N(N+1)}{2}$ variances and covariances (the second moments).

4 Capital Asset Pricing Model (CAPM)

Definition

A model used to determine a “theoretically appropriate” required rate of return for an asset, based on:

- The asset’s sensitivity to non-diversifiable (systematic) risk, measured by β .
- The expected excess return of the market.
- A risk-free rate.

Assumptions

- All investors are price-takers.
- All investors measure returns over one period.

- No short-selling constraints.
- Investors can borrow or lend at a single risk-free rate without restriction.
- All investors are mean-variance optimizers.
- All investors share the same beliefs about means, variances, and covariances.

Market Portfolio

- The market portfolio is a value-weighted index of all risky assets in proportion to their market values.
- Its return is given by:

$$R_m = \sum_{i=1}^N \frac{V_i}{V} R_i,$$

where V_i is the market capitalization of asset i and $V = \sum_{i=1}^N V_i$.

Optimal Risky (Tangency) Portfolio

Under mean-variance optimization, all investors hold the same risky portfolio p , where

$$R_p = \sum_{i=1}^N w_{pi} R_i,$$

and $\sum_i w_{pi} = 1$. If an investor wants a complete portfolio with risk-free asset f , its weights must satisfy

$$w_f + \sum_i w_i = 1, \quad \text{where } w_i = w_{pi} (1 - w_f).$$

5 Statement of CAPM & Implications

Theorem #1: Mutual Fund Theorem

- All investors hold a combination of the risk-free asset and the *same* risky portfolio p .
- The ratios w_i/w_j are the same for all investors, i.e. they hold risky assets in the same proportions.

Theorem #2: In Equilibrium, the Market Portfolio Is the Optimal Risky Portfolio

Sketch of the proof:

- Each investor's demand for a stock i is $w_i \times W$, where W is the investor's wealth. Summing across investors yields a total demand that must match the total supply (the market-cap V_i).
- Under equilibrium, we show that $w_{pi} = \frac{V_i}{V}$, implying R_p is effectively R_m . Therefore, in equilibrium, **the market portfolio is the same as the tangency portfolio** and is mean-variance efficient.

Capital Market Line (CML)

The CML is the special Capital Allocation Line (CAL) where the risky portfolio is the market portfolio. The theoretical implication is that one cannot do better (in mean-variance sense) than combining the risk-free asset with the market. Beating the market would imply private information or an arbitrage possibility, contradicting the assumptions of CAPM.

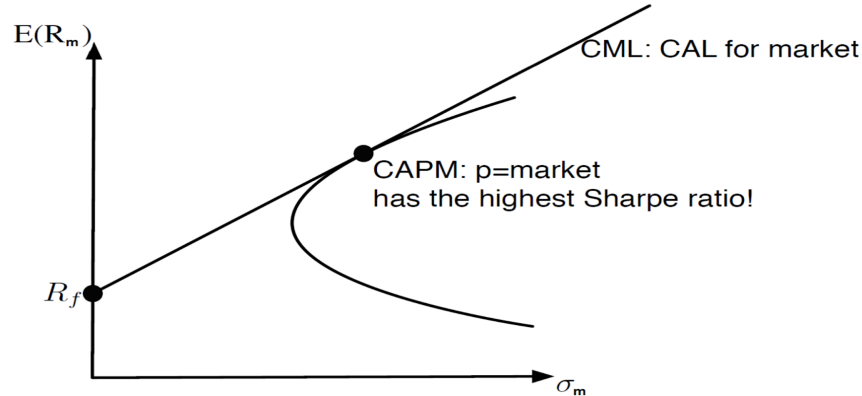


Figure 1: Capital Market Line

6 Intuition for CAPM Equilibrium

- Good news about an asset i leads to a re-estimation of its mean and covariance, causing investors to shift weight toward i .
- The price of i then increases, reducing its expected return until equilibrium is restored.

7 CAPM: Stronger Mutual Fund Theorem

- The efficient portfolios are simply combinations of the risk-free asset and the market portfolio (no need to do one's own mean-variance analysis).
- CAPM implies that market returns must satisfy the Beta Representation Theorem.

7.1 Violation of CAPM

When $\alpha_i \neq 0$, we REJECT the CAPM, meaning that there is an alternative portfolio that achieves a higher mean-variance combination than the market portfolio. Therefore, the market portfolio is no longer the optimal for all investors.

8 Alpha, Beta, and CAPM

Regression Model

$$R_{it} - R_f = \alpha_i + \beta_{iw} (R_{wt} - R_f) + \varepsilon_{it}.$$

Here, the residual ε_{it} is assumed to be conditional mean-independent (satisfies SLR.4). Under CAPM validity, the beta representation theorem holds and the OLS estimator is unbiased and consistent:

$$\beta_{iw} = \frac{\text{Cov}(R_i, R_w)}{\text{Var}(R_w)}.$$

Taking expectations of the above regression yields:

$$\mathbb{E}[R_{it}] - R_f = \alpha_i + \beta_{iw} (\mathbb{E}[R_{wt}] - R_f).$$

CAPM requires that $\alpha_i = 0$ for all i , so assets on the efficient frontier exhibit “zero alpha” (Jensen’s alpha).

9 Security Market Line (SML)

$$\mathbb{E}[R_i] = R_f + \beta_{im} (\mathbb{E}[R_m] - R_f).$$

This line in the mean–beta plane is called the SML. An asset’s alpha is its vertical distance from this line. If $\beta_{im} = 1$, then $\mathbb{E}[R_i] = \mathbb{E}[R_m]$ (meaning the asset *is* the market portfolio in risk-exposure terms).

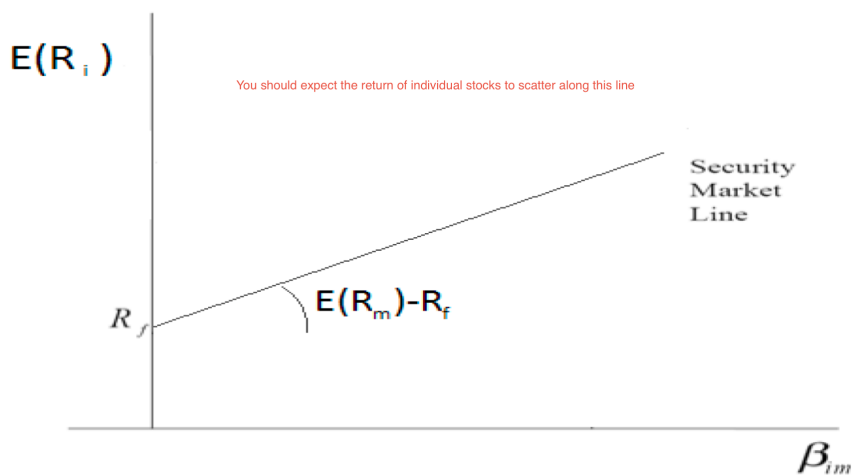


Figure 2: Security Market Line

Breaking Down Beta

Recall:

$$\rho_{im} = \frac{\text{Cov}(R_i, R_m)}{\sigma_i \sigma_m}, \quad \text{thus} \quad \beta_{im} = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)} = \rho_{im} \frac{\sigma_i}{\sigma_m}.$$

Hence,

$$\mathbb{E}[R_i] = R_f + \rho_{im} \frac{\sigma_i}{\sigma_m} (\mathbb{E}[R_m] - R_f).$$

Therefore,

- Higher ρ_{im} , lower diversification value, higher expected return $\mathbb{E}[R_i]$ needed.
- Higher $\frac{\sigma_i}{\sigma_m}$, asset i is riskier to the market. Higher expected return $\mathbb{E}[R_i]$ needed.

10 Empirical Research on CAPM

Cross-Sectional Approach

We test whether β explains average returns:

$$\mathbb{E}[R_i] - R_f = \beta_{im} (\mathbb{E}[R_m] - R_f).$$

1. First, estimate betas for each i using time-series regressions:

$$R_{i,t} - R_{f,t} = \alpha_i + \beta_i (R_{m,t} - R_{f,t}) + \epsilon_{i,t}.$$

2. Next, check if

$$\bar{R}_i - R_f = \gamma_0 + \gamma_1 \beta_i + \epsilon_i$$

holds for all i . If CAPM holds, then $\gamma_0 \approx 0$ and $\gamma_1 \approx \bar{R}_m - R_f$.

Common caveats include measurement error bias and the need to use grouping portfolios.

Time-Series Approach

Similar idea but one regresses each asset's excess return on the market's excess return to test whether α_i is zero.

11 Testing with Cross-Sectional Approach

- Sorting stocks by their betas and seeing if the resulting portfolios lie on the SML reveals “low-reward-for-beta”. Higher-beta stocks do have higher expected returns (positive slope), but not as high as CAPM predicts (the slope is shallower).

The question “Why can't one exploit this discrepancy by shorting high-beta stocks and buying low-beta stocks?” is generally answered by borrowing constraints that prevent taking large short positions.

12 Black CAPM (With Borrowing Constraints)

- Fischer Black introduced a version of CAPM where one cannot borrow at the risk-free rate, $w_f \geq 0$. Hence no pure risk-free borrowing, and investors must remain on the pure risky efficient frontier.
- When borrowing at riskless rate is prohibited, investors no longer optimize on CAL. Instead, they optimize along the curved risky frontier.
- The market portfolio still lies on the efficient frontier but not necessarily the same tangency portfolio as in standard CAPM.

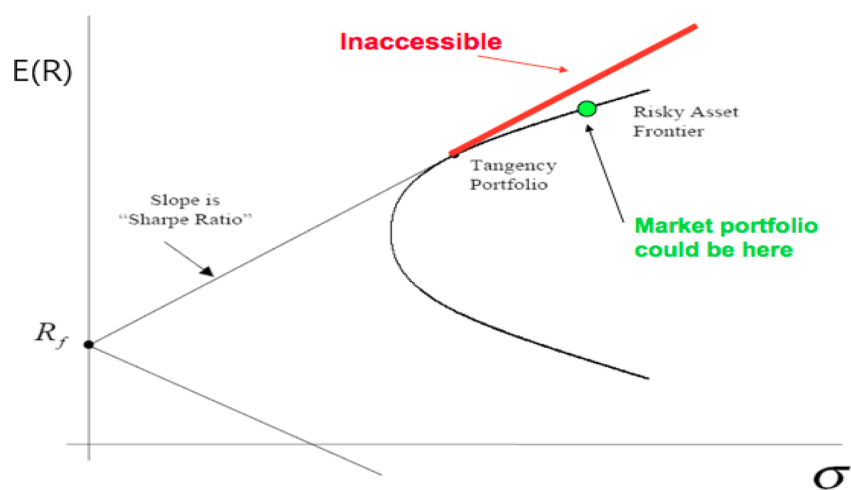


Figure 3: Black CAPM

Beta Representation for Black CAPM

Define an efficient zero-beta portfolio z uncorrelated with the market. Then:

$$\mathbb{E}[R_i] - \mathbb{E}[R_z] = \beta_{im} (\mathbb{E}[R_m] - \mathbb{E}[R_z]),$$

where $\mathbb{E}[R_z]$ acts as a “shadow risk-free rate” (larger than the actual R_f).

Lower Reward for Beta

If $\mathbb{E}[R_z] > R_f$, then

$$\mathbb{E}[R_m] - \mathbb{E}[R_z] < \mathbb{E}[R_m] - R_f,$$

meaning the slope of the “empirical” SML becomes flatter than the standard CAPM prediction. Borrowing constraints force investors to hold some high-beta stocks even if they are not sufficiently rewarded, thereby lowering the overall slope.

Figure 2 -- Average Annualized Monthly Return vs Beta for Value Weight Portfolios Formed on Prior Beta, 1928-2003

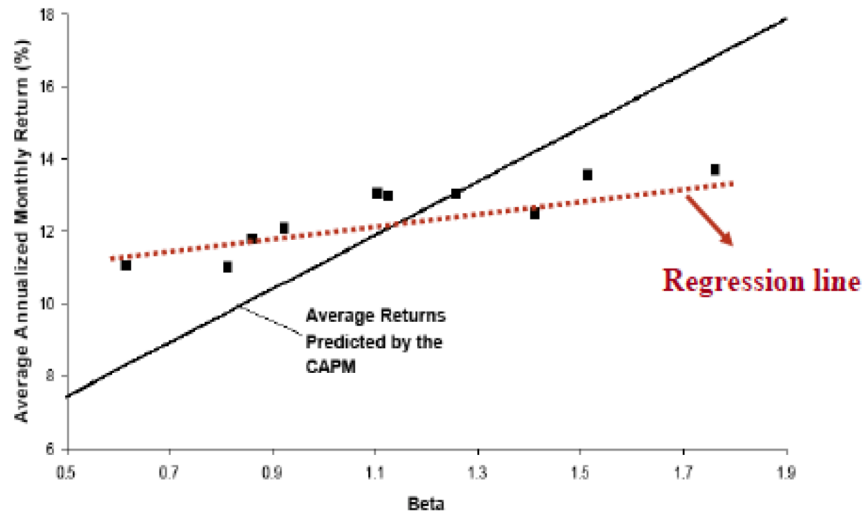


Figure 4: Lower reward for Beta

Conclusion

This thorough set of notes on CAPM and Beta Representation covers:

- The Beta Representation Theorem and its central role in pricing risky assets.
- Problems of mean-variance optimization (short-sample estimation errors, large parameter requirements, singular matrices).
- Core CAPM assumptions, derivations, and the equilibrium argument that the market portfolio is mean-variance efficient.
- Empirical findings, including the “low reward for beta” phenomenon, and how borrowing constraints (Black CAPM) explain flatter slopes.

All important formulae appear exactly as in the original lecture notes. Where pictures appeared in the original document, a “% *Insert Picture Here*” comment has been placed.

ECON0048 Lecture 8: APT & SDF & Multi-factor Models

Ambrose W

1. Market Model (Single-Index Model)

1.1 Regression Setup

We start with the following regression (“you just regress it, don’t ask”):

$$R_{it} - R_f = \alpha_i + \beta_{im} (R_{mt} - R_f) + \epsilon_{it},$$

where $(R_{mt} - R_f)$ is the **market excess return**, also referred to as the “market factor.”

1.2 OLS Parameters

From basic regression analysis using ordinary least squares (OLS), the following always hold:

$$\beta_{im} = \frac{\text{Cov}(R_{it}, R_{mt})}{\text{Var}(R_{mt})}.$$

Under the zero conditional-mean assumption (SLR.4),

$$\mathbb{E}[(R_{mt} - R_f) \epsilon_{it}] = \mathbb{E}[\epsilon_{it}] = 0.$$

2. Assumptions of APT

2.1 Uncorrelated Residual Across Stocks

$$\mathbb{E}[\epsilon_{it} \epsilon_{jt}] = 0, \quad i \neq j.$$

Therefore, the residual risk ϵ_{it} of any stock is purely *idiosyncratic* (diversifiable). Under these assumptions, it is easier to estimate covariances for mean-variance analysis:

$$\text{Cov}(R_{it}, R_{jt}) = \beta_{im} \beta_{jm} \sigma_m^2.$$

3. Arbitrage Pricing Theory (APT)

3.1 Zero Alpha and Well-Diversified Portfolios

If many assets are available, we should expect α_i to be small in absolute value. A key result: If a portfolio is well-diversified and ϵ_{it} are uncorrelated across stocks, then that portfolio's alpha is zero under no-arbitrage. The risk is fully diversifiable, so only covariance with the market is rewarded.

3.1.1 Proof Outline

Consider a portfolio p of N assets with return:

$$R_{pt} - R_f = \alpha_p + \beta_{pm} (R_{mt} - R_f) + \epsilon_{pt},$$

where

$$\alpha_p = \sum_{i=1}^N w_i \alpha_i, \quad \beta_{pm} = \sum_{i=1}^N w_i \beta_{im}, \quad \epsilon_{pt} = \sum_{i=1}^N w_i \epsilon_{it}.$$

Due to diversification,

$$\text{Var}(\epsilon_{pt}) = \text{Var}\left(\sum_{i=1}^N w_i \epsilon_{it}\right) \approx 0$$

for a well-diversified portfolio. Hence, $\epsilon_{pt} \simeq 0$ by SLR.4 and $R_{pt} - R_f \approx \alpha_p + \beta_{pm}(R_{mt} - R_f)$. Under a no-arbitrage assumption, α_p must be 0, since otherwise one could construct a near-arbitrage strategy.

3.1.2 Near-Arbitrage Construction

If $\alpha_p > 0$ and $\beta_{pm} < 1$, then:

$$R_{pt} + R_f (\beta_{pm} - 1) - \beta_{pm} R_{mt} \approx \alpha_p.$$

There is no initial investment, and the payoff is essentially risk-free profit. Hence no-arbitrage forces $\alpha_p = 0$.

3.2 Implications of APT

- Some individual α_i can be slightly mispriced. APT allows that.
- There cannot be *many* such mispriced assets with large nonzero alpha, or else arbitrage would appear.
- Zero alpha for a portfolio under APT is a weaker statement than CAPM's condition of zero alpha for *every* single asset.
- APT requires *no arbitrage* plus *uncorrelated residuals*, without needing full equilibrium or mean-variance optimization.

3.3 APT Predictions

- We derived a “beta pricing relation” purely from no-arbitrage arguments. APT DOES NOT require mean-variance optimization.
- The “market” can be any broadly diversified portfolio with uncorrelated residual risks, not necessarily the entire market portfolio from CAPM.

4. Multifactor Model

A single factor may not fully account for all correlations among individual stocks, so the model generalizes to K factors. This can yield more stable (time-invariant) β estimates. Typically $K = 4$ is enough in practice.

4.1 Factor Portfolio Method

Assume K *portfolios* that capture K underlying sources of risk and that errors across assets remain uncorrelated:

$$R_{it} - R_f = \alpha_i + \sum_{k=1}^K \beta_{ik} (R_{kt} - R_f) + \epsilon_{it}.$$

Most risky assets then have $\alpha_i = 0$. Including more and more factors is not always helpful, and a modest K often suffices.

4.2 Factor Method with Observed Factors

One can include actual variables (e.g. inflation, unemployment rates) in the regression. Let F_t be a factor with mean zero or consider $\tilde{F}_t = F_t - \mathbb{E}[F_t]$, so:

$$R_{it} - R_f = \mathbb{E}[R_{it} - R_f] + \beta_{iF} \tilde{F}_t + \epsilon_{it}.$$

For a well-diversified portfolio p ,

$$\mathbb{E}[R_{pt} - R_f] = \beta_{pF} \lambda = \sum_{k=1}^K \beta_{pF_k} \lambda_k.$$

where β_{pF} is a vector of quantity of risks and λ is a vector of risk prices.

One-Factor Case ($K = 1$) Proof Sketch

Regress two assets ($i = 1, 2$) on the factor \tilde{F}_t . Solving by substitution and applying no-arbitrage, one obtains:

$$\mathbb{E}[R_{1t} - R_f] = \lambda \beta_{1F}, \quad \mathbb{E}[R_{2t} - R_f] = \lambda \beta_{2F},$$

with

$$\lambda = \frac{\mathbb{E}[R_{1t} - R_f]}{\beta_{1F}}.$$

5. How to Pick Factors

- A broad market index is always included.
- The theory is only useful if $K \ll N$. Searching for many factors by a computer-based correlation hunt is not ideal when N is large.
- Macroeconomic factors (inflation, GDP, unemployment) sometimes perform poorly compared to factors like SMB or HML.

6. CAPM and SDF Approach

6.1 Stochastic Discount Factor (SDF) Setup

Recall:

$$p_i = \mathbb{E}[M \cdot X_i],$$

where M is the stochastic discount factor, X_i is the payoff of asset i , and p_i is its price. Dividing by the price,

$$1 = \mathbb{E}[M(1 + R_i)].$$

This holds for all assets, including the risk-free asset and market portfolio.

6.2 Theorem: CAPM Beta Pricing from a Linear SDF

Suppose $M = 1 - b R_m$. Then it implies:

$$\mathbb{E}[R_i - R_f] = \beta_i \mathbb{E}[R_m - R_f].$$

(The converse is also true: having the CAPM beta relation is equivalent to M being linear in R_m .)

6.3 Proof Outline

$$\mathbb{E}[M(1 + R_i)] = \mathbb{E}[M(1 + R_f)] \implies \mathbb{E}[MR_i] - R_f \mathbb{E}[M] = 0.$$

Using covariance properties, substituting $M = 1 - b R_m$, and comparing reward/risk across different assets, one obtains the usual CAPM form:

$$\mathbb{E}[R_i - R_f] = \beta_{im} \mathbb{E}[R_m - R_f].$$

0.1 6.4 Interpretation

- Under SDF being negatively related with the market return, then we get a positive excess return for any asset R_i that is positively related with the market.

$$\mathbb{E}[R_i - R_f] = \beta_{im} \mathbb{E}[R_m - R_f] = \rho_{im} \sigma_i S_m$$

This also explains why gold has lower returns than stocks despite similar volatility: it is negatively correlated with the market.

- CAPM follows the assumption that investors' MU declines linearly when the market goes up

7. Fama and French 4 Factor Model

7.1 Four Main Factors

1. Market Factor (MKT).
2. SMB (small minus big) – formed by shorting large-cap and going long small-cap.
3. HML (high minus low) – formed by shorting low book-to-market stocks and going long high book-to-market.
4. MOM (momentum) – formed by shorting past losers and going long past winners over the past year.

7.2 Rationale

- **Rational justification:**

- SMB reveals hidden wealth in private businesses.
- HML may reflect changing GNP and labor income that is not captured in a firm's book value.

- **Behavioral view:**

- Investors irrationally load up on growth stocks.
- Underperformance news can shift emphasis back to “value” stocks, creating the HML factor premium.

8. Final Remarks

- Arbitrage Pricing Theory (APT) uses few assumptions (no-arbitrage + uncorrelated errors across assets) to derive approximate pricing relations. Yet it does not require full equilibrium or mean-variance optimization.
- CAPM can be seen as a special case when the SDF is linearly related to the market's return.
- Multifactor models extend the idea to multiple systematic risk sources, and the Fama-French framework is a commonly used empirical specification.

ECON0048 Lecture 9.2: LIE & DDM & GGM

Ambrose W

1 Law of Iterated Expectations (LIE)

Definition and Basic Idea

$$\mathbb{E}[Y | X] = \mathbb{E}[\mathbb{E}[Y | X, Z] | X] \quad \text{or} \quad \mathbb{E}[\mathbb{E}[Y | X, Z] - Y | X] = 0 \quad \text{or} \quad \mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y | X]].$$

This states that conditioning on a larger information set must be consistent with any forecast based on a smaller set. Equivalently, the expectation made one day must equal the average of tomorrow's expectation once tomorrow's new data are considered.

LIE Over Time

Consider Y conditional on current and lagged values of X_t, X_{t-1}, \dots :

$$\mathbb{E}_t[Y] = \mathbb{E}[Y | X_t, X_{t-1}, \dots].$$

It follows that

$$\mathbb{E}_t[Y] = \mathbb{E}_t[\mathbb{E}_{t-1}[Y]] \implies \mathbb{E}_t[\mathbb{E}_{t-1}[Y] - Y] = 0.$$

Proof Sketch: If I_t denotes today's information (like X_t, X_{t-1}, \dots) and I_{t+1} denotes tomorrow's, then

$$\mathbb{E}_t[Y] = \mathbb{E}[Y | I_t] = \mathbb{E}[\mathbb{E}[Y | I_{t+1}] | I_t] = \mathbb{E}_t[\mathbb{E}_{t+1}[Y]].$$

Rearranging implies

$$\mathbb{E}_t[\mathbb{E}_{t+1}[Y] - Y] = 0.$$

2 Connecting Fundamental Value and Unpredictable Returns via LIE

Suppose an asset has a fundamental payoff X at a distant future date. Under a risk-neutral measure (or suitable SDF logic), the price equals the "fundamental value." In particular,

$$p_{n,t} = \mathbb{E}_t[X_n].$$

If additional information arrives next period, one writes

$$p_{n,t-1} = \mathbb{E}_{t-1}[X_n],$$

which changes if new information is used. By the Law of Iterated Expectations,

$$\mathbb{E}_t[p_{n,t-1}] - p_{n,t} = 0 \implies \mathbb{E}_t[p_{n,t-1}] = p_{n,t}.$$

Hence, price changes are unpredictable in an efficient market, where all public information is already incorporated. The best predictor for tomorrow's price is today's price itself.

Risk Adjustment and Time Discounting

This same principle continues to hold if the asset price is adjusted for discounting and risk via an appropriate stochastic discount factor (SDF).

Implication

In an efficient market, there are essentially two drivers of price movement:

- Normal returns demanded by investors.
- Arrival of new information.

3 Event Studies

Definition

Event studies measure the cumulative abnormal return (CAR) around announcements of corporate events, such as:

- Stock splits,
- Dividend or earning announcements,
- Mergers, takeovers, acquisitions,
- Changes in capital structure.

Abnormal return is the price fluctuation specifically attributed to the event, above or below what would be considered “normal.”

Procedure

1. Measure abnormal return by subtracting an estimate of “normal” returns from the observed return.
2. Collect these abnormal returns for a window of days around the event.
3. Average the abnormal returns across firms with similar events.
4. Cumulate the resulting values to obtain the CAR.

Some Possible CAR Patterns

Suppose there is new arrival of information at Day 0, the Top-Left figure aligns with Efficient Market Hypothesis

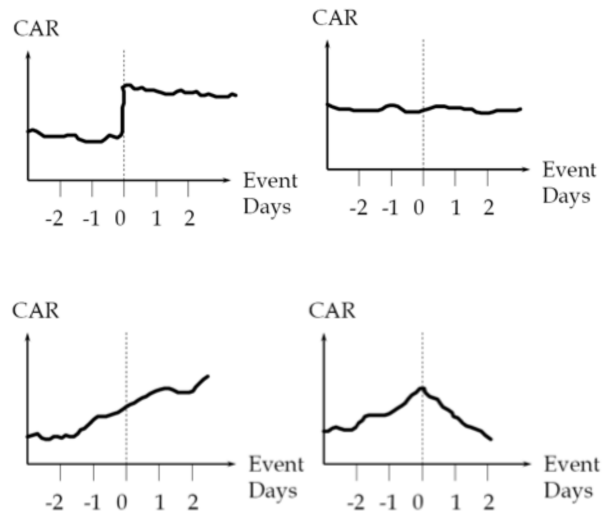


Figure 1: CAR of announcements

Examples in the notes show:

- A rapid adjustment on day 0 (consistent with the Efficient Market Hypothesis).
- No price impact if the information is unimportant.
- A gradual price response if information gets integrated slowly.

Empirical results generally show rapid price adjustment to news, though prices sometimes begin moving before a public event (semi-strong efficiency).

Example: Take-over Attempts

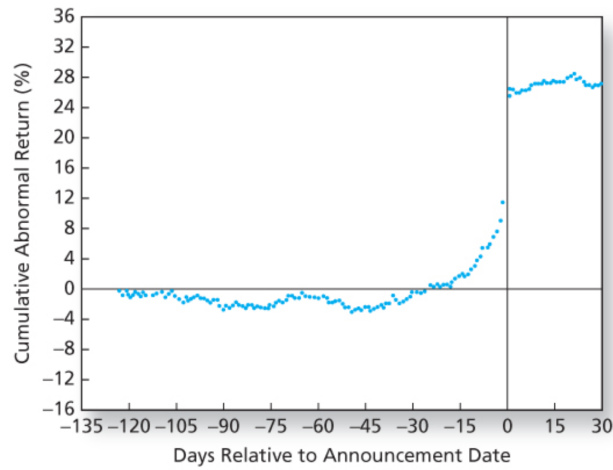


Figure 2: Response to takeover announcements

A common finding is a gradual price increase right before takeover announcements, suggesting semi-strong efficiency (some information leaks or is inferred by markets beforehand).

Earning Announcements

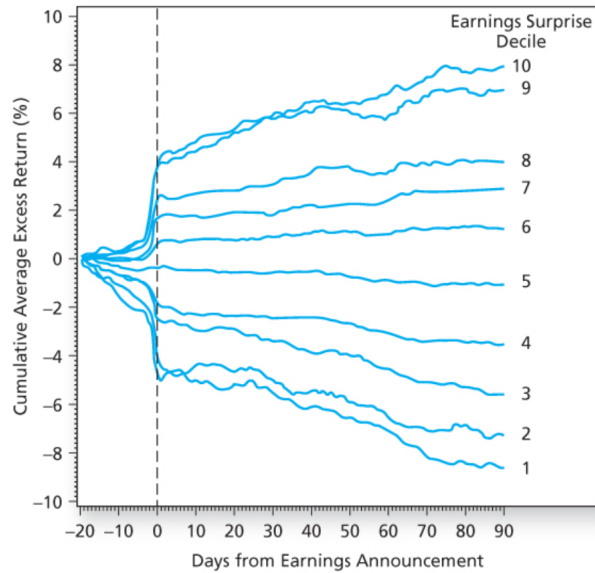


Figure 3: A 1982 study of CARs in response to earning announcements.

There used to be a post-earnings-announcement drift, which was exploited by certain quantitative strategies. However, that anomaly largely disappeared in modern markets.

4 Are Markets Efficient?

Recall from Lecture 9.1

1. There is some return predictability from past returns (though this might be subject to the joint-hypothesis critique).
2. Mutual fund managers have some skill, but on average (net of expenses) they do not beat the market.
3. Event studies show that price *does* respond to new information.

Conclusion

The EMH is not entirely correct in its pure form but holds partially. Only superior information or skill can consistently yield outperformance.

5 Dividend Discount Model (DDM)

Difference Equation

Assume an asset has a constant expected return R over time:

$$\mathbb{E}_t[R_{t+1}] = \mathbb{E}[R_{t+1} | I_t] = R.$$

Let D_t be the dividend and P_t the price. The gross return is

$$1 + R = \mathbb{E}_t[1 + R_{t+1}] = \mathbb{E}_t\left(\frac{D_{t+1} + P_{t+1}}{P_t}\right).$$

Multiplying both sides by P_t and dividing by $(1 + R)$ gives:

$$P_t = \mathbb{E}_t\left(\frac{D_{t+1} + P_{t+1}}{1 + R}\right).$$

This difference equation links P_t to P_{t+1} .

Solving Forward

Similarly,

$$P_{t+1} = \mathbb{E}_{t+1}\left(\frac{D_{t+2} + P_{t+2}}{1 + R}\right).$$

Substitute this back:

$$P_t = \mathbb{E}_t\left(\frac{D_{t+1}}{1 + R}\right) + \mathbb{E}_t\left[\mathbb{E}_{t+1}\left(\frac{D_{t+2} + P_{t+2}}{(1 + R)^2}\right)\right] = \mathbb{E}_t\left(\frac{D_{t+1}}{1 + R}\right) + \mathbb{E}_t\left(\frac{D_{t+2} + P_{t+2}}{(1 + R)^2}\right).$$

Iterating leads to:

$$P_t = \mathbb{E}_t \left(\frac{D_{t+1}}{1+R} + \frac{D_{t+2}}{(1+R)^2} + \frac{P_{t+2}}{(1+R)^2} \right).$$

Working K steps forward,

$$P_t = \mathbb{E}_t \left(\sum_{i=1}^K \frac{D_{t+i}}{(1+R)^i} \right) + \mathbb{E}_t \left(\frac{P_{t+K}}{(1+R)^K} \right).$$

Assuming no infinite-price bubble forms (so the term with P_{t+K} vanishes as $K \rightarrow \infty$),

$$P_t = \mathbb{E}_t \left(\sum_{i=1}^{\infty} \frac{D_{t+i}}{(1+R)^i} \right).$$

Conclusions of DDM

- The price is the expected discounted value of all future dividends into infinity — the asset's fundamental value.
- The discount factor $(1+R)$ reflects both time value and risk adjustments, consistent with market efficiency assumptions in deriving it.

6 Gordon Growth Model

Assumptions

1. Dividends grow at a constant rate G .
2. Next period's dividend is known one period in advance: $\mathbb{E}_t[D_{t+1}] = D_{t+1}$.
3. Dividend growth (i.i.d.) is not serially correlated.

Future Dividend in Terms of Next Period's Dividend

$$\mathbb{E}_t[D_{t+1+i}] = (1+G)^i D_{t+1}.$$

Using DDM

$$P_t = \mathbb{E}_t \left(\sum_{i=1}^{\infty} \frac{D_{t+i}}{(1+R)^i} \right) = \mathbb{E}_t \left(\frac{D_{t+1}}{1+R} + \frac{(1+G)D_{t+1}}{(1+R)^2} + \dots \right).$$

This becomes a geometric series:

$$P_t = \left(\frac{1}{1+R} \right) D_{t+1} \sum_{i=0}^{\infty} \left(\frac{1+G}{1+R} \right)^i = D_{t+1} \frac{1}{1+R} \frac{1}{1 - \frac{1+G}{1+R}} = \frac{D_{t+1}}{R-G}.$$

Equivalent Formulae

$$\frac{P_t}{D_{t+1}} = \frac{1}{R - G}, \quad \frac{D_{t+1}}{P_t} = R - G, \quad \text{and} \quad R = \frac{D_{t+1}}{P_t} + G.$$

A commonly cited long-run average from Siegel is $R = 7\%$, with $\frac{D_{t+1}}{P_t} = 4.5\%$, and $G = 2.5\%$.

Further Assumption & Results

Assume that dividend growth $\frac{D_{t+1}}{D_t}$ is identically and independently distributed (*i.i.d.*). Then:

$$\text{Var} \left(\frac{D_{t+2}}{D_{t+1}} \right) = \text{Var} \left(\frac{D_{t+1}}{D_t} \right) \quad \text{by } i.i.d.$$

The following results hold:

- $\sigma \left(\frac{D_t}{P_t} \right) = \sigma \left(\frac{D_{t+1}}{P_t}, \frac{D_t}{D_{t+1}} \right) = (\text{risk} - G) \sigma \left(\frac{D_t}{D_{t+1}} \right)$
- $\rho \left(\frac{D_t}{P_t}, \frac{D_{t-1}}{P_{t-1}} \right) = 0$
- $\sigma \left(\frac{D_{t+1}}{D_t} \right) = \sigma \left(\frac{P_{t+1}}{P_t} \right)$
- $\sigma(R_{t+1}) = \sigma \left(\frac{P_{t+1} + D_{t+1}}{P_t} \right) = \sigma \left(\frac{D_{t+2}}{D_{t+1}} \right) = \sigma \left(\frac{D_{t+1}}{D_t} \right)$

7 Gordon Growth Model and the U.S. Stock Market

Shifts in Long-Run Ratios

Dividends-to-Price (D/P) has fallen from 4.5% to about 2%. Possible explanations include:

- Higher steady-state G from higher productivity,
- Increased share repurchases reducing outstanding shares and boosting per-share dividends,
- Lower risk-free rates prompting investors to accept lower required returns.

Productivity Growth

Popular in the 1990s but less so in the following decades due to observed productivity data.

Share Repurchase

Repurchases reduce the number of shares, thus raising dividends *per share* and effectively increasing G . Compensation structures rewarding higher share prices also encourage repurchases. This phenomenon may explain about 1% decline in D/P .

Monetary Policy

As risk-free interest rates declined over the years, investors required a lower excess return on risky assets, pushing stock valuations up and reducing D/P .

ECON0048 Lecture 8.1: Efficient Market Hypothesis

Ambrose W

1 Introduction and Key Concepts

Market efficiency, also known as informational efficiency, is defined (Fama, 1970) as:

“Capital market is efficient when security prices at any time 'fully reflect' all available information.”

Keywords: Information, Price-setting, Predictability of Returns.

Implication: Superior information cannot be used to beat the market.

2 Differences to Other Concepts

2.1 Market Efficiency \neq Mean-Variance Efficiency

Mean-variance efficiency is concerned with optimizing the risk-return trade-off. In contrast, market efficiency focuses solely on the way information is reflected in security prices.

2.2 Market Efficiency \neq Pareto Efficiency

While market efficiency is a necessary condition for Pareto efficiency, Pareto optimality also requires the optimal allocation of resources. Market efficiency alone does not guarantee Pareto efficiency.

3 Testing the Efficient Market Hypothesis (EMH)

Testing whether prices reflect all available information can be decomposed into three forms based on the type of information:

3.1 Forms of EMH

1. Weak Form: Technical analysis won't work

- Information used: Historical prices (past return, trading volume).
- Implication: Current price/return should not be predicted by or correlated with past prices/returns.

2. Semi-Strong Form: Fundamental analysis won't work

- Information used: Publicly available information beyond historical prices (fundamental analysis) or private information that is very costly to obtain.

3. Strong Form: Private information won't work

- Information used: Private information (performance analysis) or scenarios in which private information is costless.

3.2 Return versus Price Views

Two distinct approaches exist when testing EMH:

- **Testing Based on Returns:**
 - Focus: Whether changes in risk-adjusted returns are unpredictable.
 - Note: Empirical work predominantly uses this approach (e.g., in the Markowitz Model and CAPM).
- **Testing Based on Prices:**
 - Focus: Whether the price level is fair given the available information.

3.3 Adjusting Returns for Risk

- **Cross-Sectional Approach:** Utilizes the CAPM.
- **Time-Series Approach:** Assumes that although risk is not time-invariant, it does not change rapidly in the short-run so that proper compensation for risk can be approximated as constant.

3.4 The Joint Hypothesis Problem

A significant issue arises when testing EMH: if risk-adjusted returns deviate from predictions, it is unclear whether the discrepancy arises from misspecification of the risk-adjustment model (e.g., omitted variables or unaccounted risk factors) or from genuine market inefficiency. For example, positive alpha in value stocks under the CAPM may be explained either by:

- **Behavioral Camp:** The CAPM correctly adjusts risk, implying that EMH is false.
- **Rational Camp:** Value stocks earn “normal” returns due to a special risk factor not captured by CAPM.

4 Testing of weak-form EMH

4.1 Serial Correlation (Autocorrelation)

Definition: The correlation of a variable with a delayed copy of itself $\rho_{R_{i,t}, R_{i,t-1}}$.

- **Positive Autocorrelation (Momentum):** High return today predicts high return tomorrow.
- **Negative Autocorrelation (Reversal):** High return today predicts low return tomorrow.

Testing Method: An auto-regressive process of order p , i.e., an AR(p) process, is employed to check for any statistically significant autocorrelation at various lags.

4.2 Serial Correlation Against Weak-Form EMH

Assume that if $R_{i,t+1}$ is the risky return tomorrow and $R_{i,t}^{\text{risk}}$ is the average risk premium, then under simplifying assumptions:

$$\text{Cov}(R_{i,t+1}, R_{i,t}) = \text{Cov}(R_t^{\text{risk}}, R_{i,t}).$$

If the compensation for risk R_t^{risk} is time-invariant, then no autocorrelation of stock returns should be observed, supporting weak-form EMH. However, when R_t^{risk} is time-variant (as is the case over a month or longer), the weak-form EMH may not hold.

Historically, the degree of autocorrelation in market returns decreased post-1970s due to the evolution of electronic trading platforms and increased information accessibility.

4.3 Momentum of Stock Returns: Definition and Characteristics

- **Momentum (MOM):** Buying recent winners and selling recent losers is a strategy believed to deliver alpha.
- Although momentum profits are generally positive on a long-term average, they can incur occasionally large losses (e.g., due to the burst of a bubble).
- Positive serial correlation in stock returns may underlie momentum.

4.4 Trading Considerations

Exploitation of momentum requires frequent trading (daily, weekly), and low transaction costs are critical for such strategies to be profitable.

4.5 Lucas' Critique

According to Lucas, once the aggregated relationship from past returns is fully utilized by market participants, the market will adjust towards weak-form EMH and the earlier serial correlation structure will dissolve.

5 Testing of semi-strong EMH: Does fund managers beat the market?

5.1 Active versus Passive Funds

Active Funds: Managed by portfolio managers who select individual stocks, factor in macroeconomic news, and use style strategies. Such funds usually exhibit a high annual turnover (approximately 100%) and incur high research and trading costs, leading to high expense ratios.

Passive Funds: Follow a mechanical approach, holding a basket of stocks selected by a third-party indexer. These funds demonstrate low turnover rates (e.g., value-weighted indices require less frequent rebalancing) and low expense ratios.

5.2 Comparative Performance

MKM Figure 4.4 indicates that from 1971 to 2011 the average index return was approximately 1% above the average fund return (net of expenses and trading costs). In fact, the index outperformed active funds in 25 of the 41 years examined.

5.3 Can Fund Managers Beat the Market?

According to Wermer (Journal of Finance, 2000), although mutual fund portfolios may beat the market in gross terms, their net returns (after expenses) underperform. There is an observed gap of approximately 2.3% between gross and net fund returns.

Table II
Mutual Fund Returns

Year	S&P 500 Return	CRSP VW Return	No.	Merged Database			
				TNA-Avg Gross Returns (%/year)	EW-Avg Gross Return (%/year)	TNA-Avg Net Return (%/year)	EW-Avg Net Return (%/year)
1975–1979	15.8	18.8	241	20.2	22.8	16.8	18.2
1980–1984	15.5	15.1	459	16.3	17.3	14.6	15.0
1985–1989	20.8	19.0	676	20.2	19.5	17.4	16.4
1990–1994	9.3	9.5	1,567	10.8	11.3	9.7	10.1
1975–1994	15.4	15.6	1,788	16.9	17.7	14.6	14.9

Figure 1: Wermer (2020) Table II

- **Cash Drag:** The cash required to execute a transaction before the selling instruction is cleared.

5.4 Testing Individual Funds' Capability

Let

$$\bar{R}_{IF} - \bar{R}_{INDEX}$$

represent the historical average excess return on an individual fund as a proxy for its capacity to outperform. A t-test is applied as follows:

$$t = \frac{\bar{R}_{IF} - \bar{R}_{INDEX}}{\sigma_{\bar{R}_{IF} - \bar{R}_{INDEX}}} \sqrt{T} \sim \mathcal{N}(0, 1),$$

which can also be interpreted as the Sharpe ratio for the fund.

5.5 Selection Bias

It must be noted that the performance evaluation of funds typically only includes those that survive over time. Funds that perform poorly may be eliminated (or sacked), thus biasing the observed performance upward. A complete analysis should consider the full population rather than only the best performers.

6 Performance Persistence of Mutual Funds

6.1 Performance Persistence

- Examines whether an out-performing fund in one period continues to outperform in subsequent periods.
- Evidence suggests that persistence among top-performing funds is partly due to momentum in their held stocks.

6.2 Skill Persistence versus Performance Persistence

- While good performance does not always persist over time, the underlying skills of proficient fund managers tend to persist.
- However, higher fees charged by skillful managers can erode net performance, as increased expenses offset gross gains.
- In equilibrium, managers (rather than investors) are compensated for their skill, although large institutions may secure lower rewards due to prestige or financial leverage.

7 Implausibility of Strong-Form EMH

7.1 Theoretical Considerations

Strong-form EMH claims that prices fully reflect all information, including private information. This is theoretically implausible when information is costly to obtain; investors will only incur these costs when the marginal benefit exceeds the marginal cost.

7.2 The Paradox of Information Gathering

An efficient market would imply that no participant needs to gather information because all information is already incorporated in prices. However, if no one gathers information, the market cannot be efficient. This paradox suggests that perfect strong-form efficiency is unattainable.

7.3 Partial Strong-Form Efficiency

Empirical evidence indicates that while research obtaining private information temporarily elevates stock prices, only a portion of such information is reflected. Consequently, there exists an equilibrium condition:

$$\text{Marginal Benefit of Obtaining New Information} = \text{Marginal Cost.}$$

7.4 Implications for Market Participation

Investors can beat the market only if they are more efficient than others in acquiring information. For those lacking such an advantage, passive investment in index funds is the recommended strategy.

7.5 Winners and Losers in Financial Markets

The so-called “Iron Law of Active Management” asserts that if some investors beat the market, there must be others who underperform. Typically:

- **Winners:** Institutional investors with superior skills.
- **Losers:** Investors with immediate liquidity needs or retail investors employing suboptimal strategies or harboring distorted beliefs.

ECON0048 Lecture 10.1 – Fixed-Income Securities and Debt Management

Ambrose W

1 Fixed-Income Basics

Definition

Fixed-income securities make fixed payments (in either nominal or real terms). Typical types of payments:

- **Zero-Coupon (Discounted) Bonds:** Pay a single amount at maturity (often normalized to \$1). Rare in practice except for very short maturities but vital for theoretical pricing.
- **Coupon Bonds:** Pay a coupon \$c each year up to *and* including maturity, then \$1 (face value) at maturity.

Markets for Bonds

- **Primary Market:** Newly issued bonds.
- **Secondary Market:** Previously issued bonds.

2 Zero-Coupon Bonds

Let

$p_{m,t}$ = Price (at period t) of a zero-coupon bond maturing in m periods with face value \$1.

Because there is a positive nominal interest rate, the price must be discounted:

$$P_{m,t} < 1.$$

3 Coupon Bonds

Let

$$C = \text{Coupon per dollar of face value (coupon rate).}$$

The price of a coupon bond maturing in m periods with coupon rate c and face value \$1 at time t is denoted:

$$P_{c,m,t}.$$

Primary Market

The coupon rate C is chosen so that the bond sells close to par ($P_{c,m,t} \approx 1$) at issuance, meaning investors pay \$1 per \$1 of face value at the start.


Secondary Market

After issuance, these bonds generally trade above or below par depending on market conditions.

4 Relationship between Coupon Bonds and Discounted Bonds

A coupon bond can be viewed as a bundle of zero-coupon bonds with varying maturities.

Example



	0	1	2	3
£100 Coupon Bonds C=0.01 per £1		£1	£1	£101
£1 1-year zero-C		£1		
£1 2-year zero-C			£1	
£101 3-year zero-C				£101
Zero-coupon Bond Portfolio		£1	£1	£101

Figure 1: Packaging coupon bonds with zero-coupon bonds

Buying \$100 of a coupon bond with $C = 0.01$ can be seen as buying:

- 1 unit of 1-year zero-coupon bond,
- 1 unit of 2-year zero-coupon bond,
- 101 units of 3-year zero-coupon bond.

Under no-arbitrage and relative pricing:

$$100P_1 + 100P_2 + 100P_3 = P_1 + P_2 + 101P_3$$

(if P_1, P_2, P_3 denote the relevant zero-coupon prices, depending on how the example is set).

Comparing Prices of Zero-Coupon Bonds

Under no-arbitrage, for instance, a 10-year zero-coupon bond issued in 2017 should be priced equivalently to a 5-year zero-coupon bond issued in 2022, assuming the same payoff date and no coupons. However, actual coupon bonds differ in price depending on the coupon rate in effect at different issuance dates.

5 Yield to Maturity (YTM)

Definition: The yearly discount rate implicit in a bond's price. It represents the average return if the bond is held to maturity.

Zero-Coupon Bond YTM

$$(1 + Y_{m,t})^m = \frac{1}{P_{m,t}} \implies P_{m,t} = \frac{1}{(1 + Y_{m,t})^m}.$$

This shows an inverse relationship between a bond's price and its YTM.

Example: A 5-year zero-coupon bond with price $P_5 = 0.95$ implies

$$(1 + Y_5)^5 = \frac{1}{0.95} \implies Y_5 = 1.03\%.$$

Log Version of the Pricing Equation

$$P_{m,t} = \frac{1}{(1 + Y_{m,t})^m}.$$

Taking logarithms:

$$\ln P_{m,t} = -m \ln(1 + Y_{m,t}),$$

denote

$$p_{m,t} = \ln(P_{m,t}), \quad y_{m,t} = \ln(1 + Y_{m,t}).$$

So,

$$p_{m,t} = -m y_{m,t}.$$

Coupon Bond YTM

The YTM of an m -period coupon bond, $Y_{c,m,t}$, is defined implicitly by:

$$P_{c,m,t} = \frac{C}{1 + Y_{c,m,t}} + \frac{C}{(1 + Y_{c,m,t})^2} + \cdots + \frac{1 + C}{(1 + Y_{c,m,t})^m}.$$

When $C = Y_{c,m,t}$, the bond sells at par ($P_{c,m,t} = 1$). If $C < Y_{c,m,t}$, it sells at a discount; if $C > Y_{c,m,t}$, it sells at a premium.

Price and YTM in the Real World

- Higher market interest rates typically imply lower bond prices.
- Longer maturity means a higher bond price at a given rate.
- Longer maturity bonds are more sensitive to interest rate changes (they face deeper discount when rates rise).

6 Yield Curve (Term Structure of Interest Rates)

A yield curve is the set of YTMs (at a given time) on zero-coupon bonds with different maturities. Longer-term rates reflect the market's expectation of future interest rate changes.

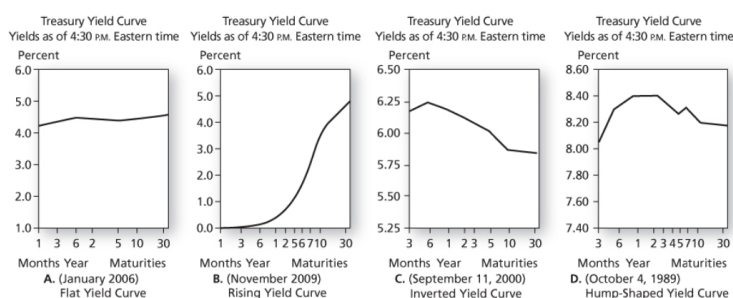


Figure 2: Yield Curves

7 Holding Period Return (HPR) for Zero-Coupon Bonds

Definition: The return on a bond if held for one year rather than to maturity.

Definition and Formula

If you hold a bond with time-to-maturity m for one year, the gross HPR is $R_{m,t-1}$:

$$1 + R_{m,t-1} = \frac{P_{m-1,t+1}}{P_{m,t}} = \frac{(1 + Y_{m-1,t+1})^{m-1}}{(1 + Y_{m,t})^m} = \frac{(1 + Y_{m,t})^m}{(1 + Y_{m-1,t+1})^{m-1}},$$

depending on the notation in the lecture. (The original lecture states it as a ratio of bond prices at different times, factoring in how maturity shortens by one year.)

Log Definition

Taking logs:

$$\ln(1 + R_{m,t-1}) = m \ln(1 + Y_{m,t}) - (m - 1) \ln(1 + Y_{m-1,t+1}),$$

defining

$$r_{m,t-1} = \ln(1 + R_{m,t-1}).$$

One obtains expressions such as

$$r_{m,t-1} = y_{m,t} - (m - 1)(y_{m-1,t+1} - y_{m,t}),$$

depending on the sign convention used.

Properties

- If the bond expires tomorrow or if the YTM is unchanged over the holding period, then the HPR equals the initial YTM.
- Decreasing YTM during the holding period increases the bond's price and thus the HPR (and vice versa).

8 Holding Period Return for Coupon Bonds

For a coupon bond with annual coupon rate $c\%$, maturity m years, and face value normalized to \$1, purchased at t and sold at $t + 1$, the HPR is:

$$R_{c,m,t-1} = \frac{(\text{Coupon received at } t + 1) + (\text{Sale price at } t + 1)}{(\text{Purchase price at } t)} - 1 = \frac{c + P_{c,m-1,t+1}}{P_{c,m,t}} - 1,$$

where $P_{c,m,t}$ is the initial purchase price and $P_{c,m-1,t+1}$ is the price (at $t + 1$) for what remains of the bond's maturity (now $m - 1$ years).

9 Government Borrowing

Governments often do not repay debt outright but roll it over by issuing new bonds:

Net Borrowing (Deficit)

An increase in debt over a certain period.

Gross Borrowing

Total amount of bonds issued:

$$\text{Gross Borrowing} = \text{Debt Maturing} + \text{Deficit}.$$

If bond markets refuse to purchase new debt in sufficient quantity, a debt crisis arises.

10 Average Maturity of Government Debt

Definition: The weighted average maturity of all outstanding bonds. Each maturity M is weighted by the total value of bonds that mature in M periods:

$$\text{AvMat} = \sum_{M=1}^m M \times w_M,$$

where

$$w_M = \frac{\text{Value of Bonds issued maturing in } M \text{ periods}}{\text{Total value of bonds issued}}.$$

Higher average maturity implies lower gross borrowing needs each year (thus lower refinancing risk).

Observation: Many countries' average maturity has risen over time, reducing the frequency with which debt must be refinanced.

11 Debt Management

Definition: The process of deciding which types of bonds a government should issue each year. Debt management agencies aim to minimize costs without increasing risk.

Dilemma

- Longer average maturity reduces refinancing risk.
- Longer average maturity typically raises borrowing costs (since term spreads tend to be positive).

Thus there is a trade-off between refinancing risk and borrowing cost.

Types of Risk

- **Interest Rate Risk:** Variability of YTM on newly issued bonds.
- **Refinancing Risk:** The risk that markets will not purchase enough new bonds (gross borrowing might exceed demand).

ECON0048 Lecture 10.2: Long-Term Bonds

Ambrose W

1 Term Structures of Interest Rates and Forward Rates

1.1 Term Structures of Interest Rates

Definition. The yield to maturity (YTM) of zero-coupon bonds can vary with different maturities:

- *Standard yield curve:* Longer maturity implies a higher YTM, denoted $Y_{m,t} > Y_{1,t}$.
- *Inverted yield curve:* Longer maturity implies a lower YTM, denoted $Y_{m,t} < Y_{1,t}$.

A yield curve conveys:

- Investor expectations about future interest rates.
- Investor expectations about future economic conditions.
- A steep yield curve might reflect fear of future government default (sovereign risk).
- An inverted yield curve can indicate a recession (because investors expect future rates to drop).

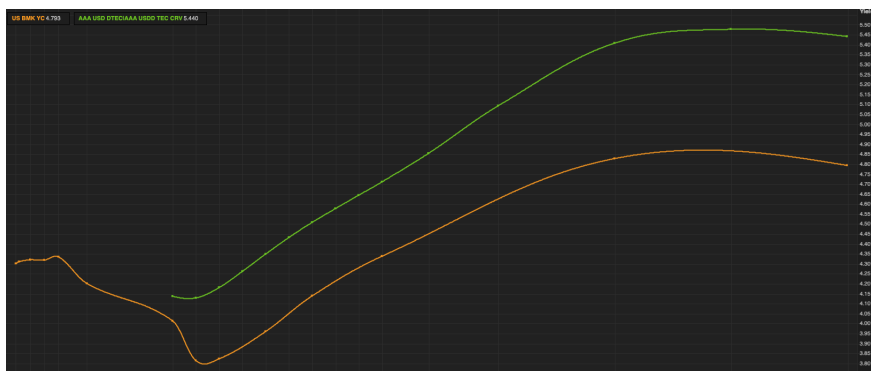


Figure 1: U.S. Gov't benchmark yield curve vs. AAA corporate (tech) yield curve

1.2 Expectation Hypothesis (EH) Overview

If the expectation hypothesis holds perfectly under no-arbitrage and no term premium, then

Long-term yield \rightarrow Market's anticipated short-term rates.

1.3 Forward Rate $F_{m,t}$

Definition. The forward rate $F_{m,t}$ is interpreted as some expectation of a bond's future yield, reflecting an interest rate between two future dates.

Level Equation.

$$1 + F_{m,t} = \frac{P_{m,t}}{P_{m+1,t}} = \frac{(1 + Y_{m+1,t})^{m+1}}{(1 + Y_{m,t})^m}.$$

Here:

- $P_{m,t}$ is the price at time t of a zero-coupon bond maturing in m periods.
- This forward rate is *not* the holding-period return (HPR); the HPR is realized *ex post*, whereas the forward rate is a theoretical construct for *ex ante* inference of future rates.

Intuition. By dividing both sides by $P_{m,t}$, each side represents an equivalent investment strategy under no-arbitrage:

$$\frac{1 + F_{m,t}}{P_{m,t}} = \frac{1}{P_{m+1,t}}.$$

- LHS: Invest at $t = 0$ and then reinvest at $t = m$.
- RHS: Directly invest from $t = 0$ until $t = m + 1$.

Log Equation. Let $f_{m,t} = 1 + F_{m,t}$. Then,

$$f_{m,t} = p_{m,t} - p_{m+1,t} = (m+1)y_{m+1,t} - m y_{m,t} = (m+1)y_{m+1,t} + y_{m,t} - (m+1)y_{m,t}.$$

Equivalently,

$$f_{m,t} = y_{m,t} + (m+1)(y_{m+1,t} - y_{m,t}).$$

2 Expectation Hypothesis (EH): Short-Term View

Definition. The proposition that the long-term rate is determined purely by current and future expected short-term rates.

Argument. The yield of a 1-year bond should be similar to the expected holding-period return (HPR) on a longer-term bond:

$$1 + Y_{1,t} \simeq \mathbb{E}_t \left[\frac{P_{m+1,t+1}}{P_{m,t}} \right].$$

Here, LHS is the gross return of the short bond, and RHS is the expected HPR of a long bond if sold after one year. By taking logs and rearranging,

$$y_{m,t} - y_{1,t} \simeq (m-1) \mathbb{E}_t [y_{m+1,t+1} - y_{m,t}].$$

Under no uncertainty and no arbitrage (or with risk neutrality and rational expectations), long-term yield equals a sequence of one-year returns.

Testing the Short-Term EH. Using a regression approach under rational expectations, one might estimate:

$$y_{m+1,t+1} - y_{m,t} = \alpha_m + \beta_m \cdot \frac{y_{m,t} - y_{1,t}}{m-1} + U_t^m.$$

EH suggests $\beta_m = 1$ for all m . Empirically, this is strongly rejected (except under extreme assumptions like perfect risk neutrality).

maturity m	1y	2y	3y	5y	7y	10y
$\widehat{\beta}^m$	-1.08	-1.42	-1.57	-1.99	-2.77	-3.73
$t - stat$	-2.14	-2.25	-2.14	-2.46	-3.28	-3.99

Figure 2: Rejection of Short-term EH

Shortcoming of the Short-Term View. Define excess HPR as:

$$xr_{m,t} = \ln\left(\frac{P_{m+1,t(1)}}{P_{m,t}}\right) - Y_{1,t}.$$

maturity m	1y	2y	3y	5y	7y	10y
mean \widehat{Y}_m	507.46	527.65	547.20	575.41	595.96	612.94
sdv $\widehat{\sigma}_{Y_m}$	327.28	321.41	314.33	299.05	286.79	271.37
sdv $\widehat{\sigma}_{xr_m}$		166.15	302.54	527.38	715.71	1011.20

Figure 3: Excess Volatility of HPR

This measure can be quite volatile, especially for longer maturity bonds. Actual bond returns from buying and selling each year (rather than holding to maturity) show higher volatility than what EH's simple equivalence suggests.

3 Expectation Hypothesis: Forward Rate View

Argument. If EH is correct, the forward rate we infer today should match the future 1-year rate realized at $t = m$. On the level scale:

$$1 + F_{m,t} = \frac{P_{m,t}}{P_{m+1,t}} \simeq \mathbb{E}_t[1 + Y_{1,t+m}].$$

On the log scale:

$$p_{m,t} - p_{m+1,t} \simeq \mathbb{E}[y_{1,t+m}],$$

and

$$f_{m,t} = y_{m,t} + (m + 1)(y_{m+1,t} - y_{m,t}) \simeq \mathbb{E}_t[y_{1,t+m}].$$

Justification by Arbitrage. One can lock in a 1-period rate at $t+m$ through a strategy of purchasing an $(m + 1)$ -period bond and simultaneously shorting an appropriate quantity of m -period bonds such that the cashflow at time t is zero. The implied interest rate locked in is:

$$1 + F_{m,t} = \frac{1}{(P_{m+1,t}/P_{m,t})}.$$

Testing the Forward View. Empirical evidence similarly rejects a perfect forward-rate EH. Bond markets exhibit lower liquidity and various constraints (borrowing constraints, OTC nature) that inhibit straightforward arbitrage.

We may also say that the **failure of EMH** \rightarrow **failure of EH**.

4 Risk in Bond Markets

4.1 Convergence & Divergence of Sovereign Debt Yields

- Different economies enact different policies after major shocks (e.g., post-Lehman).
- Default risk (e.g., Greek bond spreads after GFC).
- Sovereign risk, role of expectations, uncertainty in bond yields.

4.2 Reasons for Yield Spread

- *Default risk*: Corporate bonds default relatively frequently and unpredictably. Sovereign default risk can vary across countries.
- *Differences in liquidity*: Some debt trades in thin markets; narrower participation means higher yield demanded.
- *Inflation risk*: Long-term nominal bonds face greater inflation uncertainty and limited frequent trading.

4.3 Bond Default Risk

For corporate bonds, there is a probability of not receiving the full face value. Let h_{t+1} be the “haircut” (percentage investment loss if default). The actual return in default can be:

$$(1 + R_f)(1 - h_{t+1}).$$

Investors demand higher R_f if h_{t+1} is higher. Even bond ratings can become self-fulfilling if a lower rating raises yields, worsening financing conditions.

Default Premium under Risk Neutrality. If $\mathbb{E}[h_{t+1}]$ is the expected fraction lost upon default, a risk-neutral investor equates expected returns between risky and riskless bonds:

$$\mathbb{E}[(1 + R_d)(1 - h_{t+1})] = 1 + R_f.$$

Solving yields:

$$1 + R_d - (1 + R_d) \mathbb{E}[h_{t+1}] = 1 + R_f.$$

That difference captures the default premium.

4.4 Liquidity Risk

Bonds often have larger bid-ask spreads when markets are thinner (i.e. low capability in absorbing high-volume trade) or dominated by buy-and-hold investors. This lesser liquidity forces investors to demand higher yields.

5 Nominal and Real Interest Rate

5.1 Fisher Equation

Using ECON0016 notation:

$$r_t = \frac{i_t - \mathbb{E}[\pi_{t+1}]}{1 + \mathbb{E}[\pi_{t+1}]} \simeq i_t - \mathbb{E}[\pi_{t+1}],$$

where r_t is the real rate, i_t the nominal rate, and π_{t+1} the expected inflation.

5.2 Fisher Hypothesis

States that the real rate is exogenous to monetary policy. Hence, an increase in expected inflation simply raises the nominal rate one-for-one:

$$i \simeq r^* + \pi_{t+1}^E.$$

5.3 Fisher Hypothesis with Taxes

$$i_t \simeq \frac{r^*}{1 - \tau} + \frac{\pi_{t+1}^E}{1 - \tau}.$$

Here, τ is the tax rate, implying the nominal rate must rise by slightly more than inflation to compensate for taxes. This version ignores modern inflation-targeting regimes and financial stability concerns.

6 Inflation, Default, and Variance of Bond Returns

6.1 Default and Inflation

A surprise inflation acts like a default on nominal bonds, effectively deflating away the bond's real value.

6.2 Inflation-Linked Bonds (TIPS)

Consider a coupon bond with face value \$1 and coupon C . An inflation-indexed bond pays real (inflation-adjusted) coupon. For example, if the coupon is

$$(\text{Coupon at } M < m) : C \times \prod_{k=1}^M (1 + \pi_k),$$

the principal at maturity similarly adjusts. The real yield of TIPS satisfies:

$$1 + r = \frac{1 + i_{\text{TIPS}}}{1 + \pi},$$

where π is the inflation rate.

Break-Even Inflation.

$$\text{Spread} = i - i_{\text{TIPS}}$$

represents a measure of expected long-term inflation plus inflation risk premium (and liquidity differences). TIPS offer long-term real-value protection, appealing for pension funds and other investors wary of nominal uncertainty.

7 Long-Term View of Expectation Hypothesis (Derivation)

7.1 Two Strategies

1. Invest \$1 in a succession of 1-year zero-coupon bonds from t through $t + m - 1$. The combined return is:

$$(1 + Y_{1,t})(1 + Y_{1,t+1}) \cdots (1 + Y_{1,t+m-1}) = \prod_{j=0}^{m-1} (1 + Y_{1,t+j}). \quad (\text{Eq. 1})$$

2. Invest \$1 in a single m -period zero-coupon bond with price $P_{m,t}$ at time t . The payoff is $\frac{1}{P_{m,t}}$.

Under *no uncertainty* and *no arbitrage*, these strategies yield the same terminal value:

$$\prod_{j=0}^{m-1} (1 + Y_{1,t+j}) = \frac{1}{P_{m,t}}.$$

7.2 Risk Neutrality and Uncertainty of Future Yield

Under risk neutrality, agents equate expected returns across strategies:

$$\mathbb{E}_t \left[\prod_{j=0}^{m-1} (1 + Y_{1,t+j}) \right] = \frac{1}{P_{m,t}}.$$

Taking logarithms:

$$m \cdot y_{m,t} = \mathbb{E}_t [y_{1,t+j}].$$

Hence:

$$y_{m,t} = \frac{1}{m} \mathbb{E}_t \left[\sum_{j=0}^{m-1} y_{1,t+j} \right]. \quad (\text{Eq. 2})$$

Rearranging to express in terms of spreads:

$$y_{m,t} - y_{1,t} = \frac{1}{m} \mathbb{E}_t [(y_{1,t+j} - y_{1,t})]. \quad (\text{Eq. 3})$$

7.3 Empirical Test of Long-Term EH

One might run an OLS regression of the form:

$$\frac{1}{m} [y_{1,t+m} - y_{1,t}] = \beta_0 + \beta_1 [y_{m,t} - y_{1,t}] + u_t.$$

If the long-term EH holds, then $\beta_1 = 1$. Empirical studies (e.g., NBER Working Paper 5031, John Y. Campbell, 1995) show the long-run EH also fails, but not as dramatically as the short-term version.

We have concluded ECON0048 here, wishing you all the best in the exam.