ECON0013 T2 Lec1: Budget Constraint & Consumer Demand

Ambrose W

1 Budget Constraint

1.1 Commodity Space

Consumers purchase non–negative bundles $\mathbf{q} = [q_1, q_2, \ldots, q_k]^\top \geq 0$ from the *consumption set*. In the two–good case $\mathbf{q} = [q_1, q_2]^\top$, the consumption set is the first quadrant of the (q_1, q_2)–plane.

1.2 Standard (Linear) Budget Set

With constant prices $\mathbf{p} = (p_1, \ldots, p_k)^\top$ and total budget y,

$$\mathbf{q} \in B \quad \Longleftrightarrow \quad \mathbf{p}^\top \mathbf{q} = \sum_i p_i q_i \leq y.$$

Two–Good Case

$$B: \ p_1 q_1 + p_2 q_2 \leq y, \qquad \text{slope} \ = -\frac{p_1}{p_2}, \qquad q_1\text{–intercept} \ = \frac{y}{p_1}, \ q_2\text{–intercept} \ = \frac{y}{p_2}.$$

- B is convex and bounded.
- Any line segment connecting two feasible bundles remains inside B.

Parameter Changes (Two Goods)

- $y \uparrow$: parallel outward shift of boundary.
- $p_1 \uparrow$: steeper slope; q_1–intercept falls.
- $p_2 \uparrow$: flatter slope; q_2–intercept falls.

1.3 Non–Linear Budget Constraints

Price or available income may change with quantity.

1.3.1 Quantity ↑, Price ↑ (Tax-Kink)

Threshold E with per–unit tax t beyond it:

$$\begin{cases} p_1q_1 + p_2q_2 \le y, & q_1 \le E, \\ p_1q_1 + p_2q_2 \le y - t\,(q_1 - E), & q_1 > E. \end{cases}$$

- Budget set remains bounded and *convex*.
- Slope becomes steeper after E; boundary is continuous.

1.3.2 Quantity ↑, Price ↑ (All Units Taxed)

$$\begin{cases} p_1q_1 + p_2q_2 \le y, & q_1 \le E, \\ (p_1 + t)q_1 + p_2q_2 \le y, & q_1 > E. \end{cases}$$

Budget set is bounded but *non–convex*; boundary has a discrete jump.

1.3.3 Quantity ↑, Price ↓ (Bulk Discount)

Analogous formulas; budget set again bounded but non–convex.

2 Marshallian (Uncompensated) Demand

Demand vector $\mathbf{q} = \mathbf{f}(y, \mathbf{p})$ gives quantities demanded as functions of total budget and prices.

2.1 Total Budget Effect

2.1.1 Income Expansion Path (IEP)

Locus of demanded bundles as y varies.

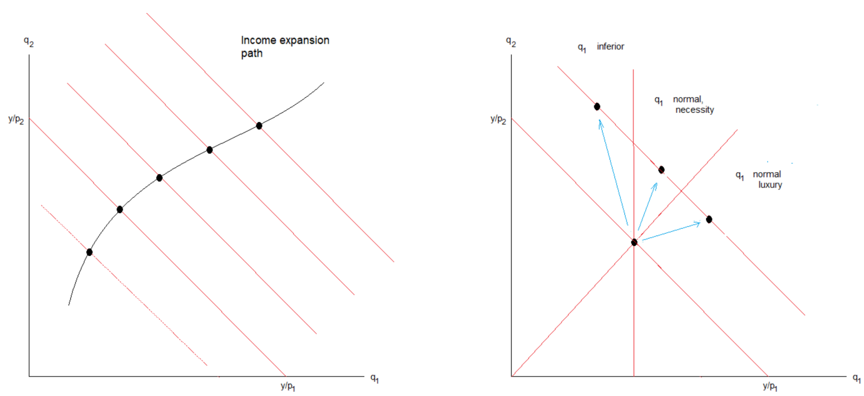

Figure 1: Income Expansion Path

NOT possible to have all goods to be inferior (i.e. move to 4th quadrant dir. will NOT occur), and NOT possible to have all goods to be necessity or luxury.

2.1.2 Engel Curves

For good i, keep $\mathbf{p}$ fixed and plot $f_i(y, \mathbf{p})$ against y.

Engel Curve Upward–sloping $\Rightarrow$ normal. Downward–sloping $\Rightarrow$ inferior.

Budget–Share Engel Curve Plot $w_i = \dfrac{p_i q_i}{y}$ against y.

- Upward–sloping $\Rightarrow$ luxury ($\varepsilon_i > 1$).
- Downward–sloping $\Rightarrow$ necessity ($\varepsilon_i < 1$).

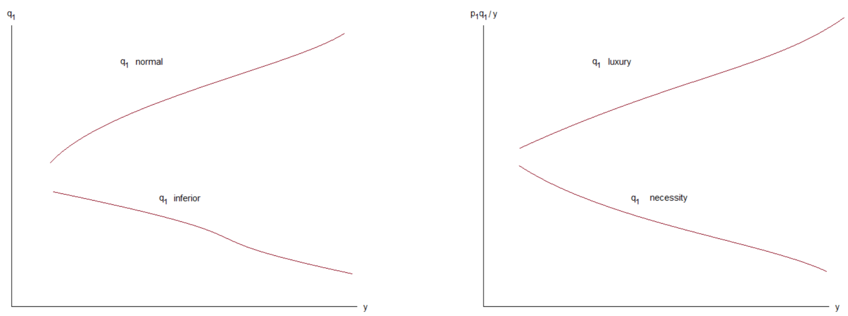

Figure 2: Engel Curve & Budget-share Engel Curve

2.1.3 Total Budget Elasticity

$$\varepsilon_i = \frac{y}{q_i} \frac{\partial f_i}{\partial y} = \frac{\partial \ln f_i}{\partial \ln y}.$$

Condition	Interpretation	ε_i
$q_i \uparrow$ when $y \uparrow$	Normal good	> 0
$q_i \downarrow$ when $y \uparrow$	Inferior good	< 0
$w_i \uparrow$ when $y \uparrow$	Luxury (income elastic)	> 1
$w_i \downarrow$ when $y \uparrow$	Necessity (income inelastic)	< 1

Table 1: Classification by Total Budget Elasticity

3 Price Effects

3.1 Demand Curve (Own-Price Variation)

Plot $f_i(y, \mathbf{p})$ as a function of p_i, holding y and other prices fixed.

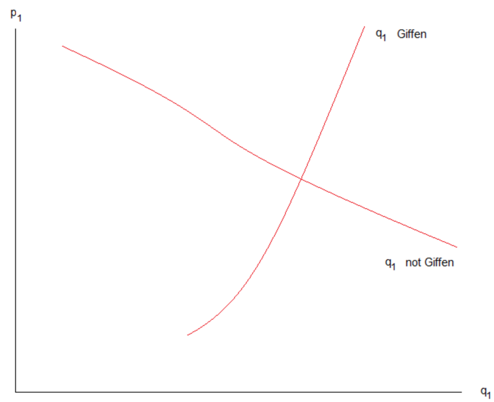

Figure 3: Demand Curve

3.1.1 Offer Curve (Two Goods)

Trace (f_1, f_2) as p_1 varies.

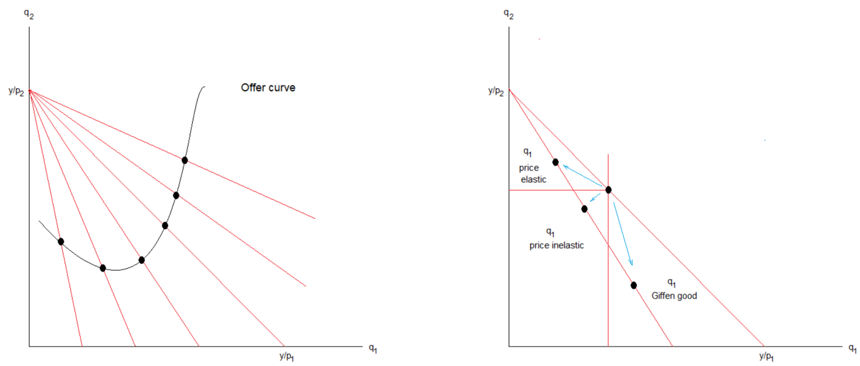

Figure 4: Offer Curve & Price Effect and PED

3.2 (Uncompensated) Own-Price Elasticity

$$\eta_{ii} = \frac{p_i}{q_i}\frac{\partial f_i}{\partial p_i} = \frac{\partial \ln f_i}{\partial \ln p_i}.$$

- $\eta_{ii} > 0 \quad \Rightarrow$ Giffen good.
- $-1 < \eta_{ii} < 0 \quad \Rightarrow$ price inelastic (budget share & expenditure rises when $p_i \uparrow$).
- $\eta_{ii} < -1 \quad \Rightarrow$ price elastic (budget share & expenditure falls when $p_i \uparrow$).

3.3 (Uncompensated) Cross-Price Elasticity

For $i \neq j$,

$$\eta_{ij} = \frac{p_j}{q_i}\frac{\partial f_i}{\partial p_j} = \frac{\partial \ln f_i}{\partial \ln p_j}.$$

- $\eta_{ij} > 0 \quad \Rightarrow$ substitute.
- $\eta_{ij} < 0 \quad \Rightarrow$ complement.

Warning: Substitutability/complementarity may not be symmetric ($\eta_{ij} \neq \eta_{ji}$).

4 Adding Up and Aggregation Restrictions

4.1 Budget Balance (Adding Up)

$$\mathbf{p}^\top \mathbf{f}(y, \mathbf{p}) = \sum_i p_i f_i(y, \mathbf{p}) = y.$$

- In a two–good world, one demand equation pins down the other.
- Serves as microfoundation for Walras' law.

4.2 Engel Aggregation

Differentiate budget balance w.r.t. y:

$$\sum_i w_i \, \varepsilon_i = 1, \qquad w_i = \frac{p_i q_i}{y}.$$

Hence, not all goods can simultaneously be inferior, luxury, or necessity.

4.3 Cournot Aggregation

Differentiate budget balance w.r.t. an arbitrary price p_j:

$$w_j + \sum_i w_i \, \eta_{ij} = 0.$$

4.4 Homogeneity of Degree Zero

Marshallian demand satisfies homogeneity of degree zero $\mathbf{f}(\lambda y, \lambda \mathbf{p}) = \mathbf{f}(y, \mathbf{p})$. Differentiating w.r.t. λ at $\lambda = 1$ yields

$$\varepsilon_i + \sum_j \eta_{ij} = 0.$$

Aggregation Condition	**Formula**
Budget balance	$\sum_i p_i f_i = y$
Engel aggregation	$\sum_i w_i \varepsilon_i = 1$
Cournot aggregation	$w_j + \sum_i w_i \eta_{ij} = 0$
Homogeneity (#0)	$\varepsilon_i + \sum_j \eta_{ij} = 0$

Table 2: Key Aggregation Restrictions

ECON0013 T2 Lec2: Revealed Preferences and Slutsky Theory

Ambrose W

1 Revealed Preferences

1.1 Definition

The revealed-preference approach infers consumer preferences from observed purchase decisions.

1.2 Direct Revealed Preference (DRP)

A bundle $\mathbf{q}_A$ is *directly revealed preferred* to bundle $\mathbf{q}_B$ (denoted $\mathbf{q}_A \, \mathcal{R}_D \, \mathbf{q}_B$) if, at prices $\mathbf{p}_A$, the consumer chooses $\mathbf{q}_A$ while $\mathbf{q}_B$ is still affordable:

$$\mathbf{p}_A^\top \mathbf{q}_A \ \geq \ \mathbf{p}_A^\top \mathbf{q}_B.$$

Strict DRP If

$$\mathbf{p}_A^\top \mathbf{q}_A \ > \ \mathbf{p}_A^\top \mathbf{q}_B,$$

then $\mathbf{q}_A$ is *strictly* DRP to $\mathbf{q}_B$.

1.3 Weak (Generalized) Axiom of Revealed Preference (WARP)

If $\mathbf{q}_A$ is strictly DRP to $\mathbf{q}_B$, no subsequent budget can exhibit $\mathbf{q}_B$ strictly DRP to $\mathbf{q}_A$.

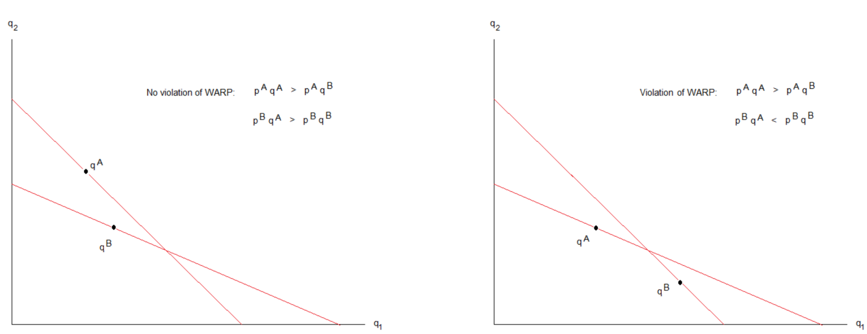

Figure 1: Violation of WARP

1.4 Indirect Revealed Preference

Chains of DRP create the broader relation $\mathcal{R}$. If $\mathbf{q}_A \mathcal{R}_D \mathbf{q}_B$ and $\mathbf{q}_B \mathcal{R}_D \mathbf{q}_C$, then $\mathbf{q}_A \mathcal{R}_D \mathbf{q}_C$.

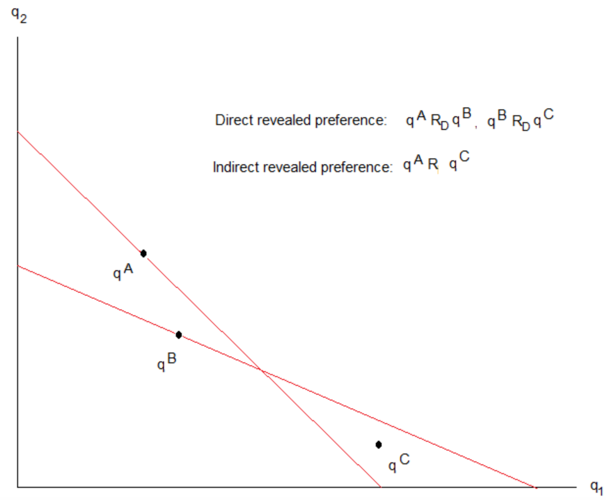

Figure 2: Indirect Revealed Preference

1.5 Strong (Generalized) Axiom of Revealed Preference (SARP)

If $\mathbf{q}_A \mathcal{R} \mathbf{q}_B$, no budget may show $\mathbf{q}_B$ strictly DRP to $\mathbf{q}_A$. In a two-good world, SARP and WARP are equivalent. However, Violation of SARP does NOT necessarily mean violation of WARP under multiple-good case.

2 Slutsky Compensation and Negativity

2.1 Slutsky Compensation

For a price rise $\mathbf{p}_A \to \mathbf{p}_B$, augment income so that the original bundle $\mathbf{q}_A$ is exactly affordable under $\mathbf{p}_B$.

$$\mathbf{p}_B^\top \mathbf{q}_A = \mathbf{p}_B'^\top \mathbf{q}_B$$

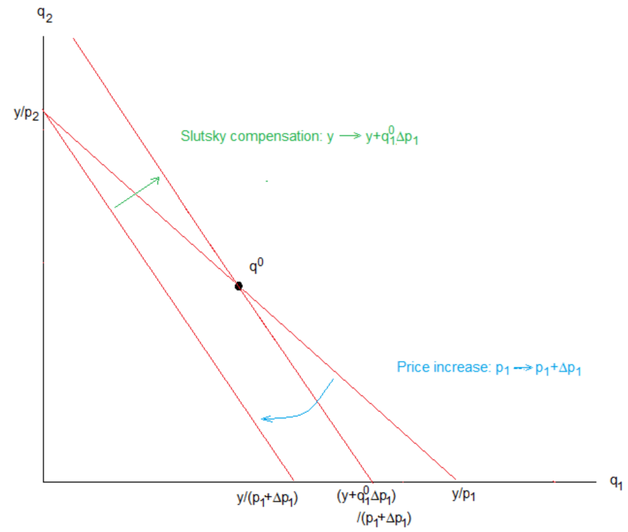

Figure 3: Slutsky Compensation

2.2 Negativity Inequality

Selecting $\mathbf{q}_B$ while $\mathbf{q}_A$ remains affordable gives

$$(\mathbf{p}_B - \mathbf{p}_A)^\top (\mathbf{q}_B - \mathbf{q}_A) \ \leq \ 0, \qquad \sum_i (p_{Bi} - p_{Ai})(q_{Bi} - q_{Ai}) < 0.$$

On net, the change in prices and the change in quantities move in opposite directions - once Slutsky compensation is applied. Thus the pure substitution (compensated) own-price effect is non-positive (i.e. consumption of a good with increased price must NOT increase)

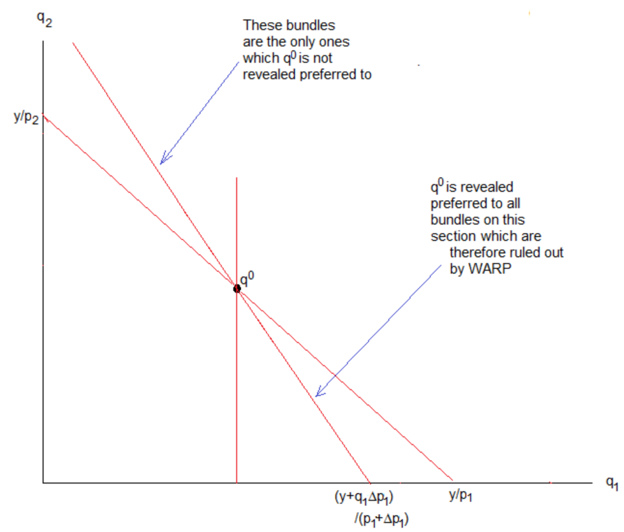

Figure 4: WARP and Slutsky Compensation

Therefore, applying Slutsky compensation (for an increase in price of one good), then the compensated own-price effect (substitution effect) MUST be negative under WARP

3 Slutsky-Compensated Demand

Define the compensated (Hicksian) demand

$$g_i(\mathbf{q}_A, \mathbf{p}) \;=\; f_i(\mathbf{p}^\top \mathbf{q}_A, \mathbf{p}),$$

with f_i the Marshallian demand.

4 Slutsky Equation

$$\frac{\partial g_i}{\partial p_j} \;=\; q_j^A \frac{\partial f_i}{\partial y} \;+\; \frac{\partial f_i}{\partial p_j} \;<\; 0.$$

Left term: *compensated price effect (substitution)*; first right term: *(minus) income effect*; second right term: *uncompensated (Marshallian) price effect*.

5 Elasticities

5.1 Compensated Price Elasticity

$$\eta^*_{ij} \;=\; \frac{\partial g_i}{\partial p_j}\frac{p_j}{g_i} \;=\; \frac{\partial \ln g_i}{\partial \ln p_j}.$$

5.2 Law of Demand

For $i = j$,

$$\eta^*_{ii} = \eta_{ii} + w_i^A \varepsilon_i,$$

where η_{ii} is the Marshallian own-price elasticity, w_i^A the budget share of good i in $\mathbf{q}_A$, and ε_i its income elasticity. Because $\eta^*_{ii} < 0$ by WARP, Marshallian demand for a *normal* good must slope downward.

Inferior and Giffen Goods If $\varepsilon_i < 0$, income and substitution effects oppose. A Giffen good requires

$$0 \;>\; \eta^*_{ii} \;>\; w_i^A \varepsilon_i,$$

so a large positive income effect (strong inferiority) outweighs the negative substitution effect.

6 Decomposing a Price Change

For $p_1 \to p_1 + \Delta p_1$ three bundles matter:

- $\mathbf{q}_0$: initial bundle,
- $\mathbf{q}_1$: new Marshallian choice (total effect),

- $\mathbf{q}_2$: Slutsky-compensated choice (substitution effect).

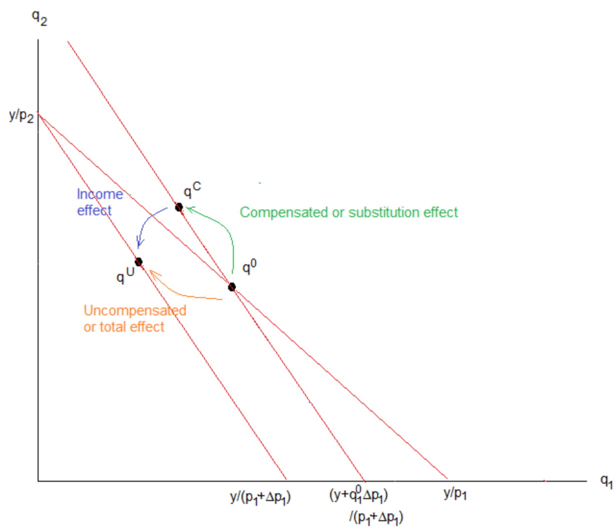

Figure 5: Price effect of Ordinary Good

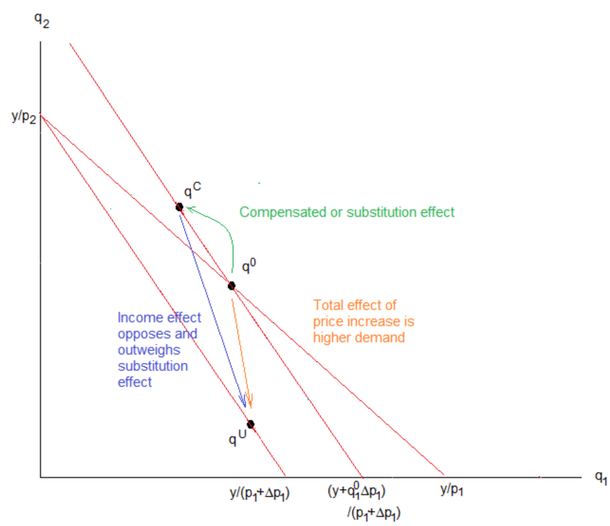

Figure 6: Price effect of Giffen Good

7 General Elasticity Formula

Multiplying the Slutsky equation by p_j/g_i and evaluating at the same bundle:

$$\boxed{\eta_{ij}^* \;=\; \eta_{ij} + w_j^A \varepsilon_i}.$$

8 Post-Compensation Negativity Conditions

With income adjusted so $p_1^\top \mathbf{q}_0 = p_1^\top \mathbf{q}_0$, WARP imposes

$$p_0^\top(\mathbf{q}_C - \mathbf{q}_0) > 0, \qquad (p_1 - p_0)^\top(\mathbf{q}_C - \mathbf{q}_0) < 0.$$

ECON0013 T2 Lec3: Consumer Preferences

Ambrose W

1. Preference Relations

1.1 Core definitions

Relation	Symbol(s)	Verbatim definition
Weak preference	$\succeq$, `\succsim`	"Bundle $\mathbf{q}_A$ is no worse than $\mathbf{q}_B$"
Strict preference	$\succ$, `\succ`	"Bundle $\mathbf{q}_A$ is strictly better than $\mathbf{q}_B$"
Indifference	$\sim$	"Consumer is equally satisfied with both bundles"
Not–indifferent	$\nsim$	—

Table 1: Verbatim preference symbols and statements.

1.2 Derived sets

$$R(\mathbf{q}_A) = \{\mathbf{q} \mid \mathbf{q} \succeq \mathbf{q}_A\}, \qquad \text{(Upper contour / weakly preferred)}$$

$$L(\mathbf{q}_A) = \{\mathbf{q} \mid \mathbf{q}_A \succeq \mathbf{q}\}, \qquad \text{(Lower contour / weakly dispreferred)}$$

$$I(\mathbf{q}_A) = \{\mathbf{q} \mid \mathbf{q} \sim \mathbf{q}_A\} = R(\mathbf{q}_A) \cap L(\mathbf{q}_A). \qquad \text{(Indifference set)}$$

2. Rationality Assumptions

(a) Completeness: For any $\mathbf{q}_A, \mathbf{q}_B$ either $\mathbf{q}_A \succeq \mathbf{q}_B$ or $\mathbf{q}_B \succeq \mathbf{q}_A$.

(b) Transitivity: For any $\mathbf{q}_A, \mathbf{q}_B, \mathbf{q}_C$, if $\mathbf{q}_A \succeq \mathbf{q}_B$ and $\mathbf{q}_B \succeq \mathbf{q}_C$ then $\mathbf{q}_A \succeq \mathbf{q}_C$.

Violations (examples kept verbatim).

- *Incomplete preferences:* $\mathbf{q}_A \succeq \mathbf{q}_B$ iff $q_{1A} \geq q_{1B} \wedge q_{2A} \geq q_{2B}$ (no ordering in quadrants II and IV).
- Criterion undefined on part of the space: $\mathbf{q}_A \succeq \mathbf{q}_B$ iff $\ln(q_{1A} - 1) + \ln(q_{2A} - 1) \geq \ln(q_{1B} - 1) + \ln(q_{2B} - 1)$.
- *Intransitive preference cycle:* $\mathbf{q}_1 \succeq \mathbf{q}_2$, $\mathbf{q}_2 \succeq \mathbf{q}_3$, yet $\mathbf{q}_3 \succ \mathbf{q}_1$.
- Majority (Condorcet) cycle: $\mathbf{q}_A \succeq \mathbf{q}_B$ iff the count of goods i with $q_{iA} > q_{iB}$ is at least as great as the count with $q_{iA} < q_{iB}$.

3. Continuity and Utility Representation

3.1 Continuity (verbatim definition)

For any $\mathbf{q}_A, \mathbf{q}_B, \mathbf{q}_C$, if $\mathbf{q}_A \succeq \mathbf{q}_B$ and $\mathbf{q}_B \succeq \mathbf{q}_C$ then there exists a bundle indifferent to $\mathbf{q}_B$ on any path joining $\mathbf{q}_A$ to $\mathbf{q}_C$. Therefore, we rule out "jumps" in preference relations.

Violation – Lexicographic preferences. $\mathbf{q}_A \succeq \mathbf{q}_B$ iff

$$q_{1A} > q_{1B} \text{ or } \big(q_{1A} = q_{1B} \ \wedge \ q_{2A} > q_{2B}\big),$$

so no bundle other than $\mathbf{q}_A$ itself is indifferent to $\mathbf{q}_A$.

3.2 Utility function

A utility function $u(\mathbf{q})$ represents preferences iff

$$u(\mathbf{q}_A) \geq u(\mathbf{q}_B) \quad \Longleftrightarrow \quad \mathbf{q}_A \succeq \mathbf{q}_B.$$

Continuity guarantees existence; lexicographic preferences admit none.

3.3 Ordinality

Any strictly increasing transformation $\phi\big(u(\mathbf{q})\big)$ preserves the order; hence u is not unique.

4. Non-satiation, Monotonicity, Convexity

Non-satiation

For every bundle, some direction yields higher utility. Indifference "sets" are therefore true curves—no thickness and no local maxima (no bliss point).

Monotonicity

"More is preferred to less" $\Rightarrow$ indifference curves slope downwards. Utility function is strictly increasing and MRS doesn't change under monotonic transformations of utility function:

$$\text{Marginal rate of substitution (MRS)} \ = \ \frac{dq_2}{dq_1} \ = \ -\frac{\partial u/\partial q_1}{\partial u/\partial q_2}.$$

Convexity

If $\mathbf{q}_A \succeq \mathbf{q}_B$ and $\lambda \in [0, 1]$, then $\lambda\mathbf{q}_A + (1-\lambda)\mathbf{q}_B \succeq \mathbf{q}_B$. Utility is *quasi-concave*:

$$\lambda u(\mathbf{q}_A) + (1-\lambda)u(\mathbf{q}_B) \ \leq \ u\big(\lambda\mathbf{q}_A + (1-\lambda)\mathbf{q}_B\big).$$

which implies the following properties:

- **Diminishing MRS:** Along an indifference curve $u = v$, rewrite $q_2(q_1)$, compute $\dfrac{d^2q_2}{dq_1^2} < 0$.
- Weakly preferred set is Convex
- Convexity of indifference curve suggests quasi-concave utility function
- Consumers always prefer to mix any two bundles between which they are indifferent (*convex combinations*).

Counter-example (verbatim). "Suppose $u(\mathbf{q}) = q_1^2 + q_2^2$; indifference curves are quarter-circles and MRS is increasing" (hence non-convex).

5. Homotheticity and Quasi-linearity

5.1 Homothetic preferences

$$\mathbf{q}_A \sim \mathbf{q}_B \implies \lambda\mathbf{q}_A \sim \lambda\mathbf{q}_B \quad (\lambda > 0).$$

Thus if $\mathbf{q}_A \succeq \mathbf{q}_B$ then $\alpha\mathbf{q}_A \succeq \alpha\mathbf{q}_B$. All points on a ray through the origin share the same MRS, written $MRS = f\left(\frac{q_2}{q_1}\right)$.

If utility is linearly *homogeneous* [$u(\lambda\mathbf{q}) = \lambda u(\mathbf{q})$], preferences are homothetic, and any increasing transformation of such a u retains homotheticity.

5.2 Quasi-linearity (with respect to good 1)

$$\mathbf{q}_A \sim \mathbf{q}_B \implies \mathbf{q}_A + \big(\lambda, 0, \ldots, 0\big) \sim \mathbf{q}_B + \big(\lambda, 0, \ldots, 0\big) \qquad \forall\lambda.$$

Hence indifference curves are parallel shifts:

$$u(q_1, q_2, \ldots) = q_1 + F(q_2, q_3, \ldots).$$

Example cited: $u(q_1, q_2) = a\,q_1 + b\ln q_2$.

7. Checklist of Equations to Memorise

1. $R(\mathbf{q}_A)$, $L(\mathbf{q}_A)$, $I(\mathbf{q}_A)$ definitions.
2. Completeness and transitivity statements.
3. Utility representation: $u(\mathbf{q}_A) \geq u(\mathbf{q}_B) \iff \mathbf{q}_A \succeq \mathbf{q}_B$.
4. MRS formula as ratio of marginal utilities.
5. Quasi-concavity inequality.
6. Homothetic scaling and homogeneity: $u(\lambda\mathbf{q}) = \lambda u(\mathbf{q})$.
7. Quasi-linear functional form $u = q_1 + F(\cdot)$.

Appendix: Examples of Preference Structures

1. Perfect Substitutes

Utility function:

$$u(\mathbf{q}) = \sum_{i=1}^{M} \alpha_i q_i$$

Marginal rate of substitution:

$$MRS = -\frac{\alpha_1}{\alpha_2} = \text{constant}.$$

Hence indifference curves are parallel straight lines.

Properties:

- Homotheticity:
$$u(\lambda \mathbf{q}) = \sum_i \alpha_i(\lambda q_i) = \lambda \sum_i \alpha_i q_i = \lambda\, u(\mathbf{q}),$$
so preferences are homothetic.

- Quasi-linearity:
$$u(q_1, q_2) = \alpha_1 q_1 + \alpha_2 q_2 = q_1 \alpha_1 + F(q_2),$$
hence quasi-linear in good 1.

- Convexity: Linear preferences yield linear indifference curves; upper contour sets are (weakly) convex.

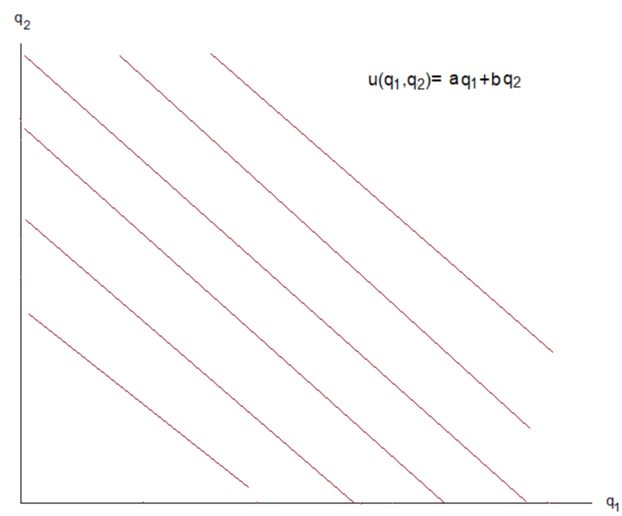

Figure 1: Indifference curves for perfect substitutes (MRS constant).

2. Perfect Complements

Utility function:
$$u(\mathbf{q}) = \min\big[\alpha_1 q_1,\, \alpha_2 q_2,\, \ldots, \alpha_M q_M\big].$$
Indifference curves are L-shaped, with kinks on rays of slope α_1/α_2.

Properties:

- Convexity: upper contour sets are convex (weakly).

- Homotheticity:
$$u(\lambda q) = \min_i\big[\alpha_i(\lambda q_i)\big] = \lambda \min_i[\alpha_i q_i] = \lambda\, u(q),$$
so homothetic.

- Quasi-linearity: not quasi-linear.

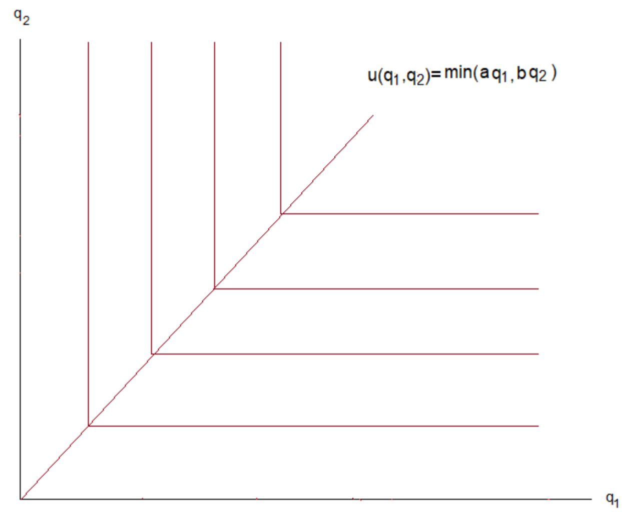

Figure 2: L-shaped indifference curves for perfect complements.

3. Cobb–Douglas Preferences

Two equivalent representations:

$$u(\mathbf{q}) = \prod_{i=1}^{M} q_i^{\alpha_i}, \quad u(\mathbf{q}) = \sum_{i=1}^{M} \alpha_i \ln q_i,$$

where $\sum_i \alpha_i = 1$ often.

Two-good case ($M = 2$):

$$u(q_1, q_2) = \alpha \ln q_1 + (1 - \alpha) \ln q_2.$$

MRS:

$$MRS = -\frac{\partial u/\partial q_1}{\partial u/\partial q_2} = -\frac{\alpha/q_1}{(1-\alpha)/q_2} = -\frac{\alpha}{1-\alpha}\frac{q_2}{q_1},$$

which is diminishing along an IC.

Budget shares: Maximisation subject to $p_1 q_1 + p_2 q_2 = y$ yields demands

$$q_i = \frac{\alpha_i y}{p_i},$$

and constant expenditure shares $w_i = \alpha_i$.

Properties:

- Convexity: strict quasi-concavity; convex upper contour.
- Homotheticity: any monotonic transform (log) of a homogeneous function is homothetic.
- Quasi-linearity: not quasi-linear.

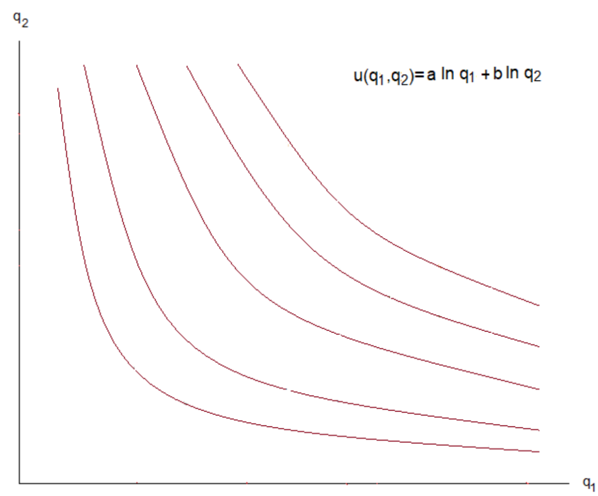

Figure 3: Smooth, convex indifference curves for Cobb–Douglas preferences.

4. Stone–Geary Preferences

Utility function:

$$u(\mathbf{q}) = \sum_{i=1}^{M} \alpha_i \ln(q_i - \gamma_i), \quad q_i > \gamma_i.$$

In two-good case:

$$MRS = -\frac{\alpha_1/(q_1 - \gamma_1)}{\alpha_2/(q_2 - \gamma_2)} = -\frac{\alpha_1}{\alpha_2}\,\frac{q_2 - \gamma_2}{q_1 - \gamma_1},$$

diminishing in q_1, q_2.

Properties:

- Convexity: strictly quasi-concave, convex upper contour.
- Homotheticity: not homothetic (due to subsistence γ_i).
- Quasi-linearity: not quasi-linear.

ECON0013 T2 Lec4 – Consumer Theory

Ambrose W

1 Consumer Optimisation

1.1 Primal problem

$$\max_{\boldsymbol{q}} u(\boldsymbol{q}) \quad \text{s.t.} \quad \boldsymbol{p}^\top \boldsymbol{q} \leq y$$

The representation of choice in this format satisfies WARP and, in fact, is equivalent to SARP under non-satiation. Homogeneity, Negativity, and Adding-up follow immediately.

1.2 Tangency and feasibility

- **Tangency condition** (monotonic, non-satiated preferences): the optimum lies where an indifference curve is tangent to the budget line.
- **Convex preferences**: uniqueness of the tangency with a linear (convex) budget set.
- **Non-negativity constraint**

$$\max_{\boldsymbol{q}} \; u(\boldsymbol{q}) \quad \text{s.t.} \quad \boldsymbol{p}^\top \boldsymbol{q} \leq y, \; \boldsymbol{q} \geq \boldsymbol{0}.$$

1. Interior solution: $\text{MRS}_{ij} = \dfrac{p_i}{p_j}$.
2. Boundary solution: one or more goods receive zero demand; the tangency may fail.

[Insert Graph: Linear budget with interior vs. boundary solutions]

Non-linear budget sets. When the feasible set is not convex, multiple tangencies can arise; utilities must be compared directly.

2 Marshallian Demand

2.1 Income expansion path (IEP)

Shifts in $(y, \boldsymbol{p})$ trace *offer curves.*

[Insert Graph: IEP and Offer Curves]

2.2 Homothetic preferences

$$f_i(y, \boldsymbol{p}) = a_i(\boldsymbol{p})\, y, \qquad i = 1, \dots, m,$$

so that each $w_i = p_i a_i(\boldsymbol{p})$ is a constant budget share.

2.3 Quasi-linearity (in good 1)

For functions $b_i(\boldsymbol{p})$, $i \geq 2$, and $B(\boldsymbol{p}) = \sum_i p_i b_i(\boldsymbol{p})$,

$$f_1(y, \boldsymbol{p}) = \frac{y - B(\boldsymbol{p})}{p_1}, \qquad f_i(y, \boldsymbol{p}) = b_i(\boldsymbol{p}),\ i > 1.$$

All income changes accrue to good 1 until a corner $\big(f_1 = 0\big)$ appears.

3 Utility Maximisation: Lagrange Method

$$\mathcal{L} = u(\boldsymbol{q}) - \lambda\big(\boldsymbol{p}^\top \boldsymbol{q} - y\big).$$

$$\frac{\partial u}{\partial q_i} - \lambda p_i = 0 \quad \forall\, i, \qquad \boldsymbol{p}^\top \boldsymbol{q} - y = 0.$$

For interior optima:

$$\mathrm{MRS}_{ij} = -\frac{\partial u/\partial q_i}{\partial u/\partial q_j} = -\frac{p_i}{p_j}.$$

Substituting the solution gives the *indirect utility function* (see Section 5).

4 Expenditure Minimisation & Hicksian Demand

4.1 Dual problem

$$\min_{\boldsymbol{q}}\ \boldsymbol{p}^\top \boldsymbol{q} \quad \text{s.t.} \quad u(\boldsymbol{q}) \geq v.$$

Lagrangian

$$\mathcal{L} = \boldsymbol{p}^\top \boldsymbol{q} - \mu\big(u(\boldsymbol{q}) - v\big).$$

FOCs (interior):

$$p_i = \mu\, \frac{\partial u}{\partial q_i}, \quad \mathrm{MRS}_{ij} = -\frac{\partial u/\partial q_i}{\partial u/\partial q_j} = -\frac{p_i}{p_j}.$$

4.2 Hicksian demand

$$q_i = g_i(v, \boldsymbol{p}).$$

Properties:

- Utility constraint: $u\big(\boldsymbol{g}(v, \boldsymbol{p})\big) \geq v$.
- Homogeneous of degree 0 in $\boldsymbol{p}$: $\boldsymbol{g}(v, \lambda \boldsymbol{p}) = \boldsymbol{g}(v, \boldsymbol{p})$.

Hicksian expenditure share:

$$w_i^g \;=\; \frac{p_i\, g_i(v,p)}{c(v,p)} \;=\; \frac{\partial \ln\bigl(c(v,p)\bigr)}{\partial \ln p_i}$$

- **Necessity:** As v increases, the compensated expenditure share w_i^g decreases.
- **Luxury:** As v increases, the compensated expenditure share w_i^g increases.
- **Neutral (unit-homothetic):** The budget share w_i^g remains constant in v.

4.3 Special cases

Homothetic preferences.

$$g_i(v,\boldsymbol{p}) = \alpha_i(\boldsymbol{p})\,\psi(v).$$

Quasi-linearity (in good 1).

$$g_1(v,\boldsymbol{p}) = \psi(v) + \beta(\boldsymbol{p}), \qquad g_i(v,\boldsymbol{p}) = b_i(\boldsymbol{p}),\ i > 1.$$

5 Indirect Utility & Expenditure Functions

5.1 Indirect utility

$$v(y,\boldsymbol{p}) = u\bigl(\boldsymbol{f}(y,\boldsymbol{p})\bigr) = \max_{\boldsymbol{q}}\bigl\{u(\boldsymbol{q}) \ :\ \boldsymbol{p}^\top\boldsymbol{q} \le y\bigr\}.$$

Monotone in y, decreasing in each p_i, and homogeneous of degree 0 in $(y,\boldsymbol{p})$.

5.2 Expenditure function

$$c(v,\boldsymbol{p}) = \boldsymbol{p}^\top\boldsymbol{g}(v,\boldsymbol{p}) = \min_{\boldsymbol{q}}\bigl\{\boldsymbol{p}^\top\boldsymbol{q} \ :\ u(\boldsymbol{q}) \ge v\bigr\}.$$

Increasing in v and every p_i; homogeneous of degree 1 in $\boldsymbol{p}$; concave in $\boldsymbol{p}$.

6 Duality Relationships

[Insert Diagram: Utility vs. Expenditure minimisation flow-chart]

$$\boldsymbol{f}\bigl(c(v,\boldsymbol{p}),\boldsymbol{p}\bigr) = \boldsymbol{g}(v,\boldsymbol{p}), \qquad \boldsymbol{f}(y,\boldsymbol{p}) = \boldsymbol{g}\bigl(v(y,\boldsymbol{p}),\boldsymbol{p}\bigr).$$
$$v\bigl(c(v,\boldsymbol{p}),\boldsymbol{p}\bigr) = v, \qquad c\bigl(v(y,\boldsymbol{p}),\boldsymbol{p}\bigr) = y.$$

7 Shephard's Lemma and Roy's Identity

$$\frac{\partial c(v,\boldsymbol{p})}{\partial p_i} = g_i(v,\boldsymbol{p}) \qquad \text{(Shephard's Lemma)}$$

$$f_i(y,\boldsymbol{p}) = -\frac{\partial v(y,\boldsymbol{p})/\partial p_i}{\partial v(y,\boldsymbol{p})/\partial y} \qquad \text{(Roy's Identity)}$$

Shephard's Lemma - Hicksian budget share

$$w_i \;=\; \frac{p_i\, g_i(v,p)}{c(v,p)} \;=\; \frac{p_i}{c(v,p)}\,\frac{\partial c(v,p)}{\partial p_i} \;=\; \frac{\partial \ln\big(c(v,p)\big)}{\partial \ln p_i}$$

8 Slutsky Equation (Hicksian form)

$$\frac{\partial g_i(v,\boldsymbol{p})}{\partial p_j} = \frac{\partial f_i(y,\boldsymbol{p})}{\partial p_j} + \frac{\partial f_i(y,\boldsymbol{p})}{\partial y}\, f_j(y,\boldsymbol{p}).$$

Own-price compensated effects satisfy

$$\frac{\partial g_i(v,\boldsymbol{p})}{\partial p_i} \leq 0.$$

Symmetry holds:

$$\frac{\partial g_i(v,\boldsymbol{p})}{\partial p_j} = \frac{\partial^2 c(v,\boldsymbol{p})}{\partial p_i\,\partial p_j} = \frac{\partial g_j(v,\boldsymbol{p})}{\partial p_i}.$$

9 Integrability Conditions

A demand system is integrable *iff*

1. Adding-up: $\boldsymbol{p}^\top \boldsymbol{f}(y,\boldsymbol{p}) = y$ and $\boldsymbol{p}^\top \boldsymbol{g}(v,\boldsymbol{p}) = c(v,\boldsymbol{p})$.
2. Homogeneity of degree 0 in $(y,\boldsymbol{p})$ or $(v,\boldsymbol{p})$.
3. Negativity: $\dfrac{\partial g_i}{\partial p_i} \leq 0$.
4. Symmetry of compensated cross-price effects.

Equivalently, no violation of SARP implies all four properties and thus integrability.

Implications.

- Existence of an indirect utility function $v(\cdot)$.
- Consistent substitution/complementarity patterns.
- Downward-sloping compensated demand curves.

10 Conceptual Map

$$\text{Expenditure function} \longrightarrow \text{Hicksian demand} \quad :: \quad \boldsymbol{q} = \boldsymbol{g}(v, \boldsymbol{p})$$
$$\text{Indirect utility} \longrightarrow \text{Marshallian demand} \quad :: \quad \boldsymbol{q} = \boldsymbol{f}(y, \boldsymbol{p})$$

ECON0013 T2 Lec5 - Consumer Welfare

Ambrose W

1 Consumer Surplus

1.1 Roy's Identity and Marginal Welfare Cost

$$\frac{\partial v}{\partial p_i} = -\frac{\partial v}{\partial y} f_i(y,p) \tag{1}$$

If the marginal utility of income $(\partial v/\partial y)$ is roughly constant, the Marshallian demand f_i proxies the marginal welfare cost of a price change. Note that $\frac{\partial v}{\partial y}$ is indeed constant if the preferences is quasi-linear.

1.2 Welfare Effect of a Price Increase $p \to p'$

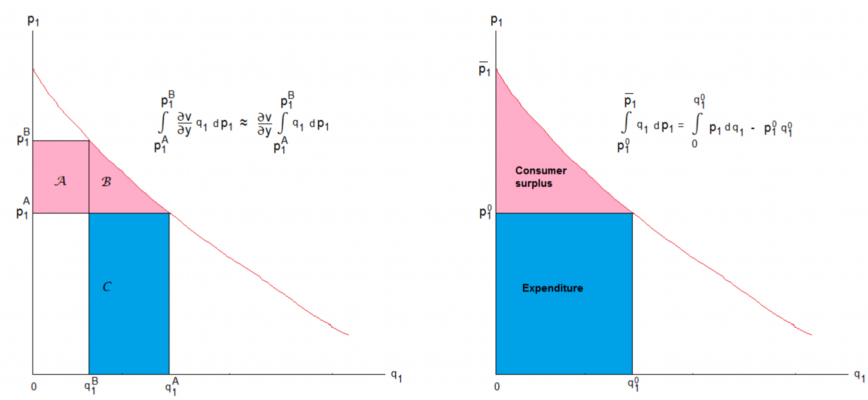

Figure 1: Welfare effect of a price increase

- **Area A**: extra payment for units still bought after the hike.
- **Area B**: loss of willingness-to-pay on units no longer bought.
- **Area C**: original expenditure at price p.

1.3 Alternative Lagrangian Interpretation

The first-order condition from $\max_q u(q)$ subject to $pq = y$ gives

$$\frac{\partial u}{\partial q} = \lambda p. \tag{2}$$

With constant λ, the accumulated "value" is the trapezoid under the demand curve; consumer surplus is this value minus actual expenditure.

2 Compensated Welfare Measures

2.1 Hicksian Demand and Shephard's Lemma

$$g_i(v,p) \;=\; \frac{\partial c(v,p)}{\partial p_i}. \tag{3}$$

Exact welfare change at fixed v: Compensating variation

$$c(v,p_B) - c(v,p_A) = \int_{p_{iA}}^{p_{iB}} \frac{\partial c(v,p)}{\partial p_i}\,dp_i = \int_{p_{iA}}^{p_{iB}} g_i(v,p)\,dp_i. \tag{4}$$

2.2 Definitions

- **Compensating Variation (CV)**: money given *after* the price change to restore the *initial* utility level.
- **Equivalent Variation (EV)**: money taken *before* the price change that leaves the consumer at the *final* utility level.
- **Consumer Surplus (CS)**: Area $A + B + C$ under the Marshallian demand curve.

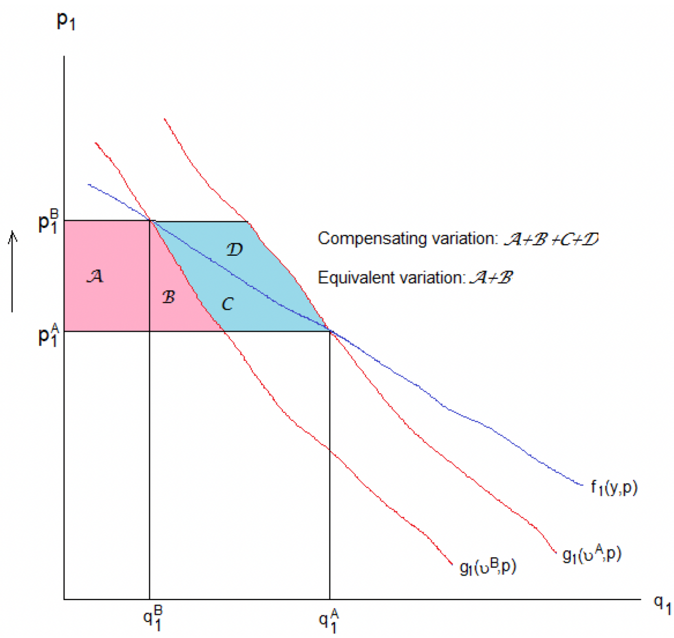

Figure 2: Hicksian welfare measures

2.3 Quasi-linear Preferences

Under quasi-linearity

$$\text{CV} = \text{EV} = \text{PS}. \tag{5}$$

(The income effect vanishes; compensated, uncompensated and Marshallian demands coincide.)

3 True Cost-of-Living Index (Konüs)

$$T(v, p_1, p_0) \;=\; \frac{c(v, p_1)}{c(v, p_0)}. \tag{6}$$

T generally depends on the utility level v, hence effect of price change tends to be different for people of different standard of living. But two special cases remove this dependence:

3.1 Proportional Price Changes

If $p_1 = \lambda p_0$ for every good,

$$c(v, \lambda p_0) = \lambda\, c(v, p_0) \quad \Longrightarrow \quad T(v, p_1, p_0) = \lambda. \tag{7}$$

Laspeyres (L) and Paasche (P) indices both equal λ.

3.2 Homothetic Preferences

With homothetic preferences

$$c(v, p) = p_i\, \alpha_i(p)\psi(v), \tag{8}$$

let $\psi(v) = v$

$$T(v, p_1, p_0) = \frac{vA(p_1)}{vA(p_0)} = \frac{A(p_1)}{A(p_0)}, \tag{9}$$

Because budget shares are the same at all standard of living v under homothetic preferences, the index is the same at all v regardless of prices

4 Approximate Cost-of-Living Indices

4.1 Laspeyres Index

$$L(q^0, p^1, p^0) = \frac{p^1 q^0}{p^0 q^0} = \sum_i w_i^0 \frac{p_i^1}{p_i^0}, \tag{10}$$

where

$$w_i^0 = \frac{p_i^0 q_i^0}{\sum_j p_j^0 q_j^0}. \tag{11}$$

Properties.

- $L < 1$ (with unchanged nominal expenditure) $\Rightarrow$ consumers cannot be worse off.
- Substitution bias: $L \geq T(v_0)$. (The base bundle need not be cost-minimal at p^1.)

4.2 Paasche Index

$$P(q^1, p^1, p^0) = \frac{p^1 q^1}{p^0 q^1} = \left[\sum_i w_i^1 \frac{p_i^0}{p_i^1} \right]^{-1}. \tag{12}$$

Properties.

- $P > 1$ (with unchanged expenditure) $\Rightarrow$ real purchasing power falls.
- Substitution bias: $P \leq T(v_1)$.

4.3 Bounds under Homothetic Preferences

With homothetic preferences

$$P \;\leq\; T \;\leq\; L. \tag{13}$$

Here T is independent of the utility level, so the inequality chain holds for all consumers.

5 Key Take-aways

- Roy's identity links Marshallian demand to marginal welfare cost (1).
- CV, EV and CS coincide under quasi-linear tastes (5).
- The Konüs index T captures the *exact* cost-of-living change (6).
- Laspeyres overestimates, Paasche underestimates T because of substitution bias (10)–(12).
- Under homothetic preferences, $P \leq T \leq L$ and all three indices are utility-invariant (13).

ECON0013 T2 Lec6 – Endowments and Labor Supply

Ambrose W

1 Endowments and the Consumer Problem

1.1 Setup

An individual is endowed with the bundle $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots)$. Preferences are $u(\boldsymbol{q})$; the budget constraint becomes

$$\max_{\boldsymbol{q}} \; u(\boldsymbol{q}) \quad \text{s.t.} \quad \boldsymbol{p}\cdot\boldsymbol{q} \; \leq y + \boldsymbol{p}'\boldsymbol{\omega},$$

where y is endowed income and $\boldsymbol{p}'\boldsymbol{\omega}$ is the value of endowed goods.

- Without y, any deviation from ω_i requires selling one good to buy another. Price changes rotate the budget line around the endowment point (ω_1, ω_2).
- With $y > 0$, the consumer may be a net *buyer* of both goods. The affordable set is the straight line joining the points $\big(\omega_1,\ \omega_2 + y/p_2\big)$ and $\big(\omega_1 + y/p_1,\ \omega_2\big)$. Its slope is $-\frac{p_1}{p_2}$.
- When price p_1 increases, the budget line rotates *clockwise* around the point $\big(\omega_1,\ \frac{y}{p_2} + \omega_2\big)$.
  - This is the point at which the consumer is neither buying nor selling good 1 whose price has changed.
  - It represents the amount they consume if unaffected by the change in price of good 1.
- When price p_2 increases, the budget line rotates *anti-clockwise* around the point $\big(\omega_1 + \frac{y}{p_1},\ \omega_2\big)$.

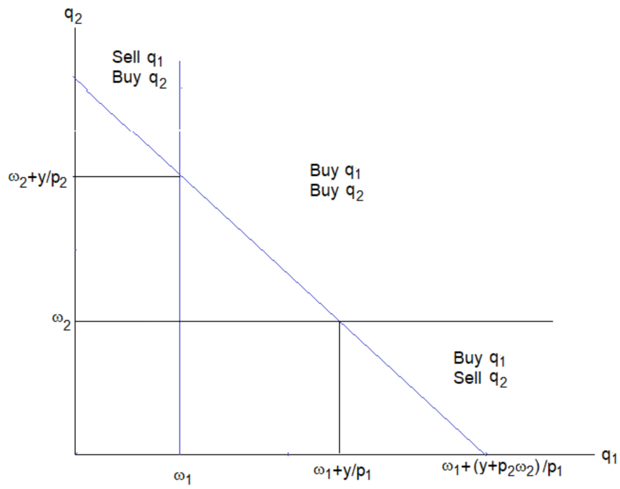

Figure 1: Budget constraint with endowments

1.2 Demand with Endowments

Gross (Marshallian) Demand

$$q_i = f_i(Y, \boldsymbol{p}), \qquad Y := y + \boldsymbol{p}'\boldsymbol{\omega}.$$

Compensated (Hicksian) Demand Unchanged because preferences and prices are unchanged.

Net Demand

$$z_i = \phi_i(y, \boldsymbol{\omega}, \boldsymbol{p}) = f_i\big(Y, \boldsymbol{p}\big) - \omega_i = q_i - \omega_i.$$

Sign convention: $z_i > 0$ (net buyer), $z_i < 0$ (net seller).

2 Comparative Statics

2.1 Change in Endowed Income y

Effect on gross demand = Effect on net demand

$$\frac{\partial q_i}{\partial y} = \frac{\partial f_i(Y, \boldsymbol{p})}{\partial Y} = \frac{\partial \phi_i(y, \boldsymbol{\omega}, \boldsymbol{p})}{\partial y}, \qquad \frac{\partial \omega_i}{\partial y} = 0.$$

2.2 Change in Endowment ω_j

The only effect of change in endowment on quantity consumed (i.e. gross demand) is the change in value of the endowment.

Note that when $i \neq j$, change in gross demand = change in net demand

$$\frac{\partial q_i}{\partial \omega_j} = \frac{\partial f_i}{\partial Y} p_j, \qquad \frac{\partial \phi_i}{\partial \omega_j} = \begin{cases} p_i \dfrac{\partial f_i}{\partial Y} - 1, & i = j, \\ p_j \dfrac{\partial f_i}{\partial Y}, & i \neq j. \end{cases}$$

2.3 Change in Prices p_j

Effect on gross demand = Price effect + Endowment income effect

$$\frac{\partial q_i}{\partial p_j} = \underbrace{\frac{\partial f_i}{\partial p_j}}_{\text{Direct price effect}} + \underbrace{\frac{\partial f_i}{\partial Y} \omega_j}_{\text{Endowment income effect}} .$$

Connection with Hicksian Demand

$$\frac{\partial q_i}{\partial p_j} = \frac{\partial g_i}{\partial p_j} - q_j \frac{\partial f_i}{\partial Y} + \frac{\partial f_i}{\partial Y} \omega_j = \frac{\partial g_i}{\partial p_j} - (q_j - \omega_j) \frac{\partial f_i}{\partial Y}.$$

where $(q_j - \omega_j)\frac{\partial f_i}{\partial Y}$ is the net demand weighted endowment income effect.

Own–price Effect ($i = j$)

$$\frac{\partial q_i}{\partial p_i} = \frac{\partial g_i}{\partial p_i} - (q_i - \omega_i) \frac{\partial f_i}{\partial Y}.$$

Note that

- $\frac{\partial g_i}{\partial p_i} < 0$ due to WARP
- $\frac{\partial f_i}{\partial Y} > 0$ if i is a normal good
- $q_i - \omega_i > 0$ if i is a net buyer or < 0 if i is a net seller

Therefore, for buyers of normal goods, demand curves still slope down due to negative income effect. For sellers of normal goods, sign of income effect is reversed (+ve income effect), therefore the direction of overall price effect & slope of demand curve is ambiguous unless you know the individual is selling or buying

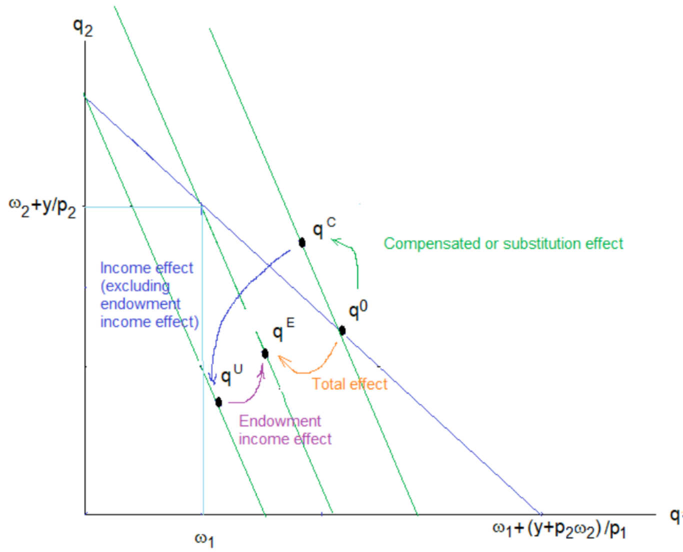

Figure 2: Endowment income effect: $p_1 \uparrow$ and net buyer of good 1

3 Revealed Preference with Endowments

- If an individual is a *seller* of good 1 and $p_1 \uparrow$, the original bundle remains affordable. By WARP, the consumer will not switch to a bundle where $z_1 > 0$; hence they remain a seller.
- Symmetrically, a *buyer* will not switch to being a seller when the price falls.
- If the price moves in an individual's favor, then they will NOT switch their position in the market [WARP]

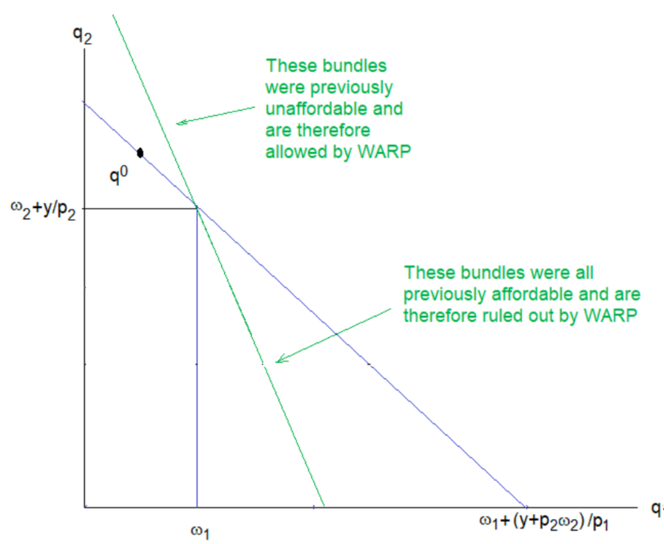

Figure 3: Revealed preference with endowments

4 Labour Supply with Endowments

4.1 Framework

Let leisure be $h = T - \ell$ and consumption be c. Utility: $u(c, h)$. Unearned income: m. Nominal wage: w. Price of consumption: p.

4.1.1 Full-income Budget Constraint

$$pc + wh \leq m + wT \equiv M.$$

Equivalent form: $pc \leq m + w\ell$.

4.2 Demand for Leisure

$$h = f\big(wT + m,\ w,\ p\big) = \phi(m, w, p) + T = g(v, w, p),$$

where v is the indirect-utility level.

Wage Effect on Leisure (Slutsky)

$$\left.\frac{\partial f}{\partial w}\right|_m = \frac{\partial \phi}{\partial w} = \frac{\partial g}{\partial w} - (h - T)\frac{\partial f}{\partial M}.$$

- If the individual *sells* labour ($h < T$) and leisure is normal, the substitution effect ($\partial g/\partial w < 0$) opposes a positive income effect, rendering the total effect of higher wage on (uncompensated) labor supply is ambiguous.

4.3 Labour-supply Functions

$$\ell = L(m, w, p) = T - h = T - f(wT + m, w, p), \qquad \ell = \Lambda(v, w, p) = T - g(v, w, p).$$

Response to Wage

$$\frac{\partial L}{\partial w} = -\frac{\partial \Lambda}{\partial w} + L\,\frac{\partial L}{\partial M}.$$

Substitution ($-\partial\Lambda/\partial w > 0$) and income effects ($< 0$) work in opposite directions; overall sign is ambiguous.

4.4 Backward-bending Labour-supply Curve

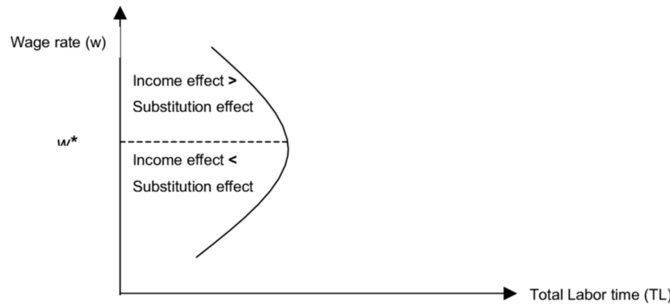

Figure 4: Backward-bending labour supply curve

- Low wage: +ve substitution effect dominates $\Rightarrow$ supply slopes upward.
- High wage: -ve income effect dominates $\Rightarrow$ supply slopes downward.

4.5 Example: Cobb–Douglas Utility

$$u(c,h) = \alpha \ln h + (1-\alpha)\ln c, \qquad wh + pc = wT + m.$$

Substituting $h = T - \ell$, $c = (w\ell + m)/p$, the optimisation condition

$$\frac{(1-\alpha)w}{w\ell + m} = \frac{\alpha}{T - \ell}$$

yields

$$\boxed{\ell = (1-\alpha)T - \frac{\alpha m}{w}}.$$

Hence labour supply is strictly increasing in w; income effects never dominate for Cobb–Douglas preferences.

Key Takeaways

1. Endowments tilt the budget line around the endowment point and introduce *net* demand.
2. A price change induces an additional *endowment–income* effect proportional to the value change of the endowment.
3. Hicksian–Marshallian decomposition extends: the net demand $q_j - \omega_j$ scales the income effect.
4. Revealed-preference restrictions imply that favourable price movements do not reverse trading positions.
5. In labour supply, wage changes alter both the relative price of leisure and the value of time endowment, giving rise to potentially backward-bending supply.

ECON0013 T2 Lec7 - Intertemporal Choice

Ambrose W

Intertemporal Choice

Model Setup

- Two periods: $t = 0, 1$
- Two consumption goods: $c_0,\ c_1$
- Endowed incomes: $y_0,\ y_1$
- Real, risk–free interest rate: r
- Period utility is additive; no uncertaintys

(a) Lifetime Budget Constraint

$$(y_0 - c_0)(1 + r) + (y_1 - c_1)\ \geq\ 0$$

Equivalently,

$$c_0 + \frac{c_1}{1 + r}\ \leq\ y_0 + \frac{y_1}{1 + r}.$$

(b) Demand for Current Consumption

$$c_0\ =\ f_0\Big(y_0 + \tfrac{y_1}{1+r},\ 1 + r\Big),$$

i.e. consumption today depends on the present–discounted value of lifetime income.

(c) Effect of a Change in the Interest Rate

Substitution effect: If $r \uparrow$, future consumption is relatively cheaper $\Rightarrow c_1 \uparrow$ (unambiguously).

Income effect:
- **Saver** (net seller at $t = 0$): $r \uparrow \Rightarrow\ c_0 \uparrow,\ c_1 \uparrow$.
- **Borrower** (net buyer at $t = 0$): $r \uparrow \Rightarrow\ c_0 \downarrow,\ c_1 \downarrow$.

(d) WARP Arguments

- A saver will not switch to borrowing if r increases.
- A borrower will not switch to saving if r decreases.

(e) Lifetime Utility

$$U(c_0, c_1) \;=\; v(c_0) \;+\; \frac{1}{1+\delta}\, v(c_1),$$

where $\delta > 0$ is the subjective discount rate (impatience).
Within–period utility $v(\cdot)$ is strictly increasing and strictly concave:

$$v'(c) > 0, \quad v''(c) < 0.$$

(f) Marginal Rate of Substitution

$$MRS \;=\; -\,\frac{v'(c_0)}{v'(c_1)}\,(1+\delta).$$

Greater $\delta \Rightarrow$ steeper intertemporal indifference curves. $\Rightarrow$ Higher impatience

(g) Intertemporal Elasticity of Substitution

$$\sigma = \frac{\partial \ln(c_1/c_0)}{\partial \ln(1+r)} \;=\; -\,\frac{\partial \ln\big(v'(c_1)/v'(c_0)\big)}{\partial \ln(1+r)}.$$

Higher σ implies lower concavity/ curvature of $v(\cdot)$.

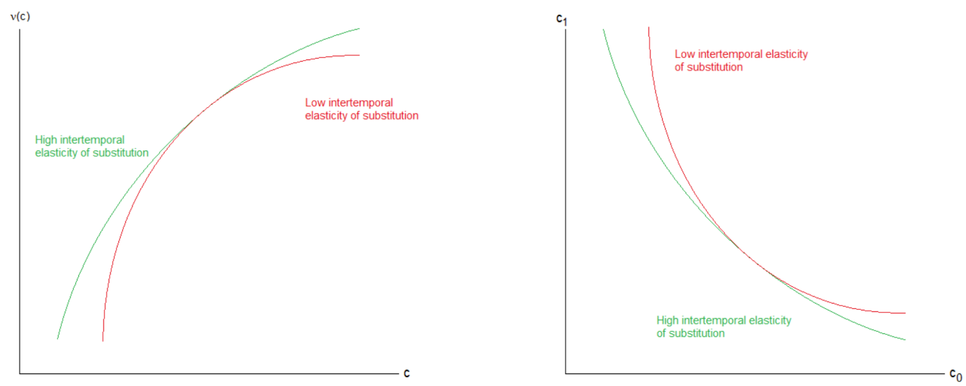

Figure 1: Left: Concavity of $v(\cdot)$ and IES. Right: Convexity of IBC

Saving and Borrowing: Optimization

(1) Problem Statement

We substitute c_1 from the IBC into the utility function:

$$\max_{c_0}\; v(c_0) \;+\; \frac{1}{1+\delta}\, v\big((y_0 - c_0)(1+r) + y_1\big).$$

(2) Consumption Euler Equation

First–order condition:

$$v'(c_0) \;-\; \frac{1}{1+\delta}\, v'(c_1)\,(1+r) \;=\; 0,$$

equivalently,

$$\frac{v'(c_0)}{v'(c_1)} \;=\; \frac{1+r}{1+\delta}.$$

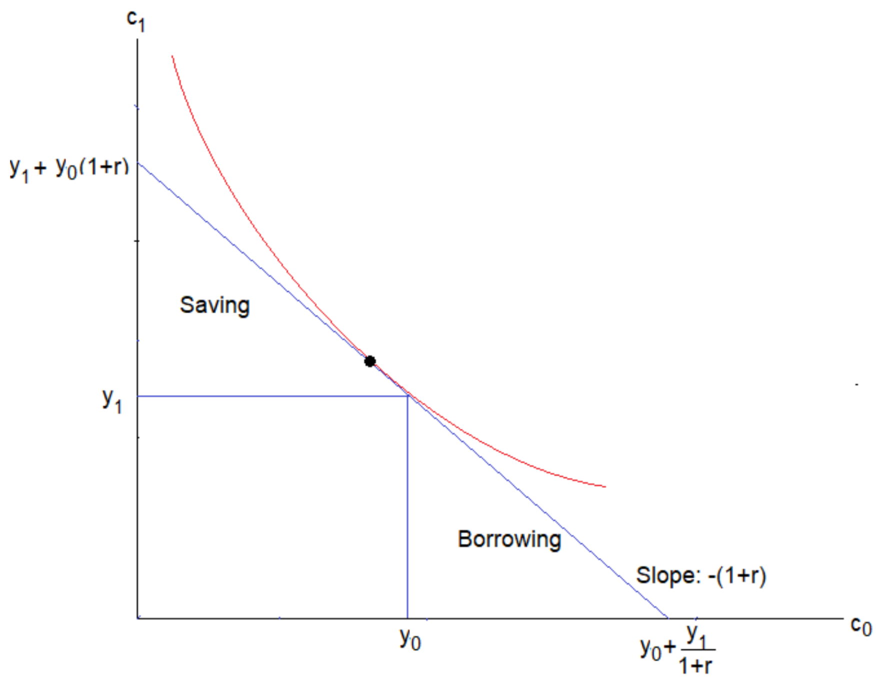

Figure 2: Intertemporal optimization

(3) Comparative Statics

(i) $r = \delta$**.**

$$v'(c_0) = v'(c_1) \implies c_0 = c_1 \quad \text{(complete smoothing)}.$$

(ii) $r > \delta$**.**

$$\frac{v'(c_0)}{v'(c_1)} = \frac{1+r}{1+\delta} > 1 \implies v'(c_0) > v'(c_1) \implies c_0 < c_1.$$

The market incentive to delay consumption dominates over the consumer's impatience.

(iii) $r < \delta$**.**

$$v'(c_0) < v'(c_1) \implies c_0 > c_1.$$

Asset Choice with a Concave Project

(1) Extended Budget Constraint

Introduce investment X with a strictly concave payoff $F(X)$, where $F'(X) > 0,\ F''(X) < 0$. Then

$$c_1 = (y_0 - c_0 - X)(1+r) \ + \ y_1 \ + \ F(X).$$

(2) Optimization Problem

$$\max_{c_0, X} \ v(c_0) \ + \ \frac{1}{1+\delta}\, v\big((y_0 - c_0 - X)(1+r) + y_1 + F(X)\big).$$

- **Consumption choice:** yields the same Euler equation as before.

$$\frac{v'(c_0)}{v'(c_1)} \ = \ \frac{1+r}{1+\delta}.$$

- **Investment choice:** Investment decision is independent to your consumption decision & intertemporal preferences.

$$-(1+r) + F'(X) = 0 \implies F'(X) = 1 + r.$$

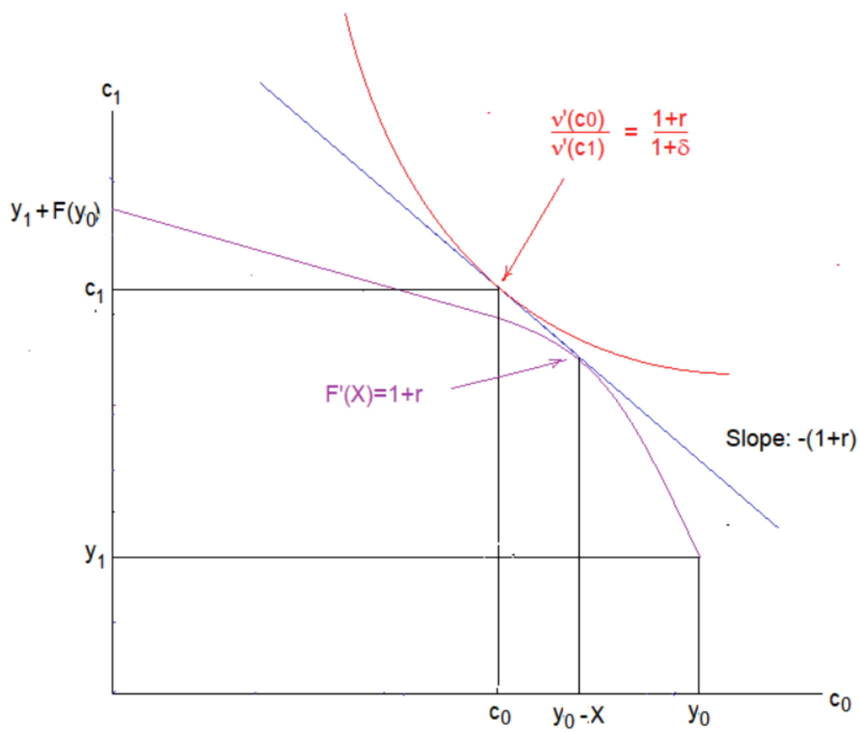

Figure 3: Intertemporal optimization with investment

(3) Comparative Statics

- If $F(\cdot)$ becomes more productive, X increases.
- Higher returns on X raise consumption in both periods (income effect).

Appendix: Constant Intertemporal Elasticity of Substitution

Preferences:

$$u(c_0, c_1) = v(c_0) \;+\; \frac{1}{1+\delta}\, v(c_1), \quad v_t(c) = \frac{1}{1+\gamma}\; c_t^{1+\gamma},\ \gamma < 0, \quad v(c_t) = \ln c_t,\ \gamma = 1.$$

Hence

$$v(c) = \frac{1}{1+\gamma}\; c^{1+\gamma}, \quad (\gamma \neq -1), \quad v(c) = \ln c \quad (\gamma = -1).$$

Euler equation:

$$\frac{c_1}{c_0} = \Big(\frac{1+r}{1+\delta}\Big)^{\frac{1}{\gamma}}.$$

Compute IES:

$$\ln\!\Big(\frac{c_1}{c_0}\Big) = -\frac{1}{\gamma}\;\ln\!\Big(\frac{1+r}{1+\delta}\Big), \quad \sigma = -\frac{1}{\gamma}.$$

Appendix: Life–Cycle Spending with Age–Dependent Needs

Stone–Geary preferences:

$$u(c_0, c_1) = \ln(c_0 - \gamma) \;+\; \frac{1}{1+\delta}\; \ln(c_1 - \gamma), \quad v(c) = \ln(c - \gamma).$$

Demand functions:

$$c_0 = \gamma \;+\; \frac{1+\delta}{2+\delta}\Big[y_0 - \gamma + \tfrac{y_1 - \gamma}{1+r}\Big],$$
$$c_1 = \gamma \;+\; \frac{1+r}{2+\delta}\Big[y_0 - \gamma + \tfrac{y_1 - \gamma}{1+r}\Big].$$

Consumption path is not flat despite $r = \delta$ if $\gamma \neq \gamma$.

ECON0013 T2 Lec8 - Choices under Uncertainty

Ambrose W

Choice under Uncertainty

1. Certainty vs. Uncertainty

- **Certainty**: smooth consumption across periods.
- **Uncertainty**: transfer resources between states of the world via insurance, gambling, fraud.

Budget Constraint

- **Certainty** determined by interest rates and returns.
- **Uncertainty** determined by insurance premium γ, betting odds z, chances of getting caught if fraud.

Preferences

In both cases, utility is additive across periods or states. Under uncertainty, within-state utility $v(\cdot)$ is concave to capture risk aversion.

2. Modelling Choices under Uncertainty

Let consumption in two states be c_0, c_1 with subjective probabilities $\Pi, 1-\Pi$. A lottery is $(c_0, c_1; \Pi)$.

Expected Utility (Von Neumann–Morgenstern)

$$\mathbb{E}[U] = \Pi\, v(c_0) + (1-\Pi)\, v(c_1).$$

3. Sure–Thing Principle & Allais Paradox

Sure–Thing Principle If a third state leaves payoffs unchanged, choice between prospects should be invariant.

Choice	0.01	0.10	0.89	Notes
A1	$1m	$1m	$1m	$1m with certainty
B1	$0	$5m	$1m	
A2	$1m	$1m	$0	$1m with 0.11 probability
B2	$0	$5m	$0	$5m with 0.10 probability

Figure 1: Payoff of Allais paradox

Allais Paradox

- **Choice 1:**

A: \$1 m for certain;   B: \$1 m w.p. 89%, \$5 m w.p. 10%, \$0 w.p. 1%.

- **Choice 2:**

A: \$1 m w.p. 11%, otherwise \$0;   B: \$5 m w.p. 10%, otherwise \$0.

Switching from A to B violates the principle since the "sure" branch is irrelevant.

4. Risk Aversion

An individual is risk-averse if for any gamble

$$\Pi\, v(c_0) + (1-\Pi)\, v(c_1) < v\big(\Pi\, c_0 + (1-\Pi)\, c_1\big),$$

with equality for risk neutrality (v linear).

5. Insurance

Asset value A, incur loss with probability Π. C_0 is the consumption if no loss, C_1 if loss.

Without Insurance

$$c_0 = A, \quad c_1 = 0.$$

With Insurance

Pay premium γK to receive K if loss:

$$c_0 = A - \gamma K, \quad c_1 = K - \gamma K = (1-\gamma)K.$$

Budget constraint:

$$(1-\gamma)\, c_0 \;+\; \gamma\, c_1 = A(1-\gamma).$$

Full insurance ($K = A$): $c_0 = c_1 = (1-\gamma)A$. Over-insurance is prohibited.

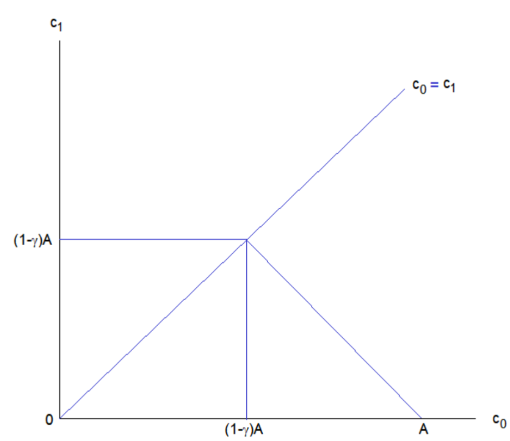

Figure 2: Budget line with insurance

Optimization

Individual choose K (how much to insure) to maximize expected utility

$$\max_K \ \Pi\, v(K - \gamma K) + (1 - \Pi)\, v(A - \gamma K),$$

First-order condition:

$$\Pi\, v'(c_1)(1 - \gamma) + (1 - \Pi)\, v'(c_0) = 0 \quad \Longrightarrow \quad \frac{v'(c_0)}{v'(c_1)} = \frac{\Pi(1 - \gamma)}{\gamma(1 - \Pi)},$$

where $c_0 = A - \gamma K$, $c_1 = (1 - \gamma)K$.

Actuarial Fairness If $\Pi = \gamma$, expected return zero. Risk-averse individuals choose full insurance ($K = A$). If $\gamma > \Pi$: under-insurance ($K < A$). Over-insurance prohibited if $\Pi > \gamma$.

6. Gambling

Wealth A, bet B, subjective win probability Π, odds z.

Odds

Based on bookmaker's belief Π^*:

$$z = \frac{1 - \Pi^*}{\Pi^*}.$$

Outcomes

$$c_0 = A - B, \quad c_1 = A + B\, z.$$

Expected utility:

$$\mathbb{E}[U] = (1 - \Pi)\, v(A - B) + \Pi\, v(A + Bz).$$

Marginal rate of substitution:

$$\text{MRS} = -\frac{(1 - \Pi)\, v'(c_0)}{\Pi\, v'(c_1)}, \quad \text{at } c_0 = c_1 : \ \text{MRS} = -\frac{1 - \Pi}{\Pi}.$$

Multiplicative Bet

Bet xA:

$$\mathbb{E}[U] = (1 - \Pi)\, v(A(1 - x)) + \Pi\, v(A(1 + zx)).$$

FOC:

$$-(1 - \Pi)\, v'(A - B) + \Pi\, z\, v'(A + Bz) = 0.$$

Rationality Constraint

Bet only if

$$\Pi\, z - (1 - \Pi) > 0 \iff z > \frac{1 - \Pi}{\Pi}.$$

Kelly Criterion (Log Utility)

For $v(c) = \ln(c)$, optimal fraction:

$$\frac{B}{A} = \frac{\Pi - \Pi^*}{1 - \Pi^*}.$$

ECON0013 T2 Lec9 - General Equilibrium

Ambrose W

1 Exchange Equilibrium

1.1 Basic Setup

- Economy with H households, n goods.
- Household h's consumption vector: $q^h = [q_1^h, ..., q_n^h]^T$.
- Household h's initial endowment vector: $\omega^h = [\omega_1^h, ..., \omega_n^h]^T$.

1.2 Feasible Allocation

An allocation $(q^1, ..., q^H)$ is feasible if total consumption equals total endowment for each good i:

$$\sum_{h=1}^{H} q_i^h = \sum_{h=1}^{H} \omega_i^h \quad \forall i = 1, ..., n$$

1.3 Edgeworth Box (2 Individuals, 2 Goods)

A graphical tool representing all feasible allocations in a 2x2 economy.

- Dimensions: Total endowment of Good 1 (width) and Good 2 (height).
- Origins: Opposite corners for individuals A and B.
- Points inside: Represent feasible allocations.
- Indifference Curves: Can be drawn for both individuals from their respective origins.
- Pareto Superior Set: The lens-shaped area between indifference curves passing through the initial endowment ω, where both individuals are better off.

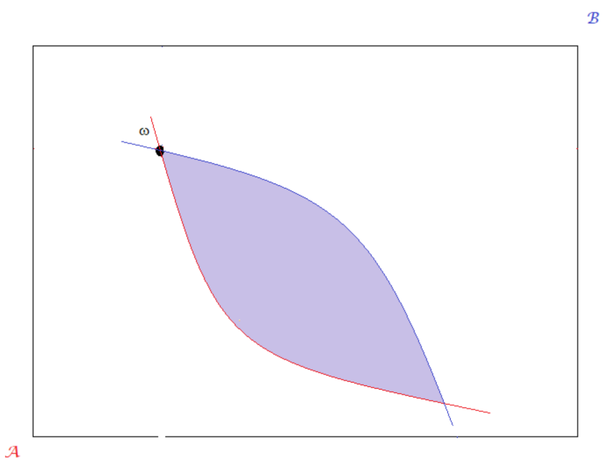

Figure 1: Basic Edgeworth Box

1.4 Trading and Equilibrium

- Given prices $p = (p_1, p_2)$, a budget line passes through ω.
- Individuals choose optimal points where their indifference curve is tangent to the budget line.
- Generally, desired trades don't match at arbitrary prices (excess supply/demand).
- **General Equilibrium (GE) / Competitive Equilibrium**: A price vector p^* and an allocation where desired trades match (optimal points coincide) and all markets clear. Graphically, indifference curves are tangent to each other and the common budget line.

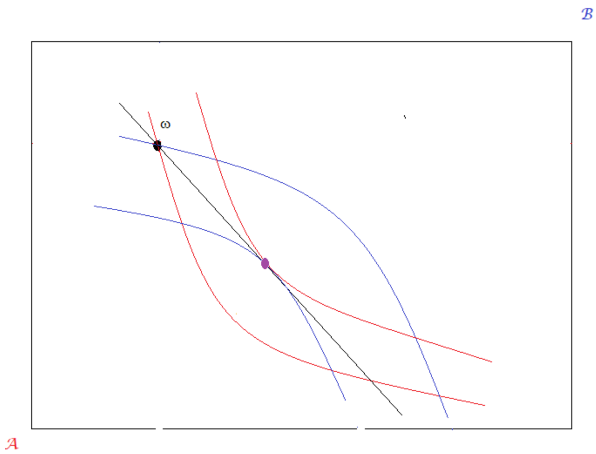

Figure 2: Edgeworth Box with General Equilibrium

2 General Equilibrium (Generalized)

2.1 Excess Demand

- Individual h's demand for good i: $q_i^h = f_i^h(p'\omega^h, p)$ (Marshallian demand).
- Individual excess demand for good i: $z_i^h = q_i^h - \omega_i^h$.
- Aggregate excess demand for good i: $Z_i(p) = \sum_h z_i^h = \sum_h q_i^h - \sum_h \omega_i^h$.

2.2 Definition of General Equilibrium

A price vector p^* is a GE price vector if aggregate excess demand is zero for all goods:

$$Z_i(p^*) = 0 \quad \forall i = 1, ..., n$$

This means all markets clear simultaneously. Only relative prices matter due to homogeneity of degree 0 of demand functions.

2.3 Walras' Law

The total value of aggregate excess demand is always zero, for any price vector p:

$$\sum_{i=1}^{n} p_i Z_i(p) = 0$$

Implication: If $n-1$ markets are in equilibrium ($Z_i(p^*) = 0$ for $n-1$ goods), the n-th market must also be in equilibrium ($Z_n(p^*) = 0$). We only need to solve for $n-1$ relative prices.

2.4 Existence, Uniqueness, Stability

- **Existence:** GE generally exists if preferences are convex and demands are continuous. Offer curves (loci of optimal bundles as prices vary) must intersect.

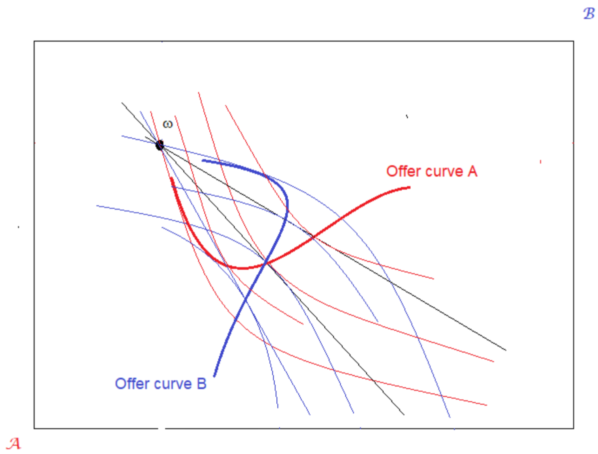

Figure 3: Offer Curve - Unique Equilibrium

- **Uniqueness:** Not guaranteed. Multiple equilibria are possible if offer curves intersect more than once.

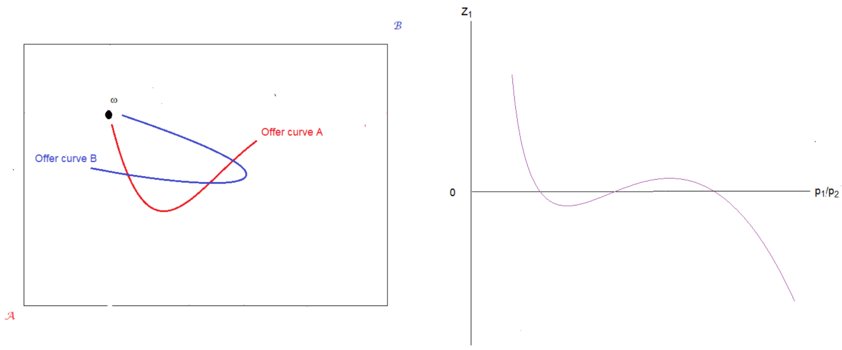

Figure 4: Offer Curve - Multiple Equilibria

- **Stability:** Not guaranteed. Depends on how prices adjust to disequilibrium (e.g., Walrasian tâtonnement: p_i rises if $Z_i > 0$, falls if $Z_i < 0$). Some equilibria may be unstable.

3 Equilibrium with Production

3.1 Firms and Production

- K firms, firm k's production plan y^k (vector: positive entries are outputs, negative are inputs).
- Technology constraint: $y^k \in Y^k$ (production possibility set). E.g., $Q \leq F(L)$.
- **Profit Maximization:** Firms choose y^k to maximize $\Pi^k = p'y^k$ subject to $y^k \in Y^k$.
- First-Order Condition (e.g., for $Q = F(L)$, price p, wage w): Maximize $pF(L) - wL \implies pF'(L) - w = 0 \implies F'(L) = w/p$ (Marginal Product = Real Wage).

3.2 Household Income and Budget Constraint

- Profits are distributed to households via ownership shares θ^{hk} ($\sum_h \theta^{hk} = 1$).
- Household h's budget constraint: $p'q^h \leq p'\omega^h + \sum_k \theta^{hk}\Pi^k = p'\left(\omega^h + \sum_k \theta^{hk} y^k\right)$.

3.3 Aggregate Excess Demand with Production

Total household demand minus total endowment minus net output from firms:

$$Z_i(p) = \sum_h q_i^h - \sum_h \omega_i^h - \sum_k y_i^k$$

3.4 Competitive Equilibrium with Production

A price vector p^* and allocation (q^{h*}, y^{k*}) such that:

1. Firms maximize profits: y^{k*} solves $\max p^{*'} y^k$ s.t. $y^k \in Y^k$.
2. Households maximize utility: q^{h*} solves $\max U^h(q^h)$ s.t. budget constraint (using p^* and $\Pi^{k*} = p^{*'} y^{k*}$).
3. Markets clear: $Z_i(p^*) = 0 \quad \forall i$. (Equivalently, $\sum_h q_i^{h*} = \sum_h \omega_i^h + \sum_k y_i^{k*}$)

Walras' Law still holds. Existence requires convex preferences and convex technology (e.g., non-increasing returns). Uniqueness and stability are not guaranteed.

4 Robinson Crusoe Economy (1 Consumer, 1 Firm)

A simple illustration of GE with production (Goods: Labor L, Consumption c).

- Firm: Uses labor L to produce c via $c = F(L)$. Maximizes profit $\Pi = pc - wL$. Chooses L^* such that $F'(L^*) = w/p$.
- Consumer: Supplies labor l, consumes c. Has preferences $U(c, \text{leisure})$. Maximizes utility subject to budget $pc \leq wl + \Pi$. Chooses l^*, c^* such that $MRS_{\text{leisure},c} = w/p$.
- Equilibrium: Occurs at a real wage $(w/p)^*$ and allocation $(L^* = l^*, c^* = Q^*)$ where:

  $$MRS_{\text{leisure},c} = F'(L^*) = (w/p)^*$$

  Graphically, the Production Possibility Frontier ($c = F(L)$) and an indifference curve are tangent. The slope of the tangent line is the equilibrium real wage $(w/p)^*$.

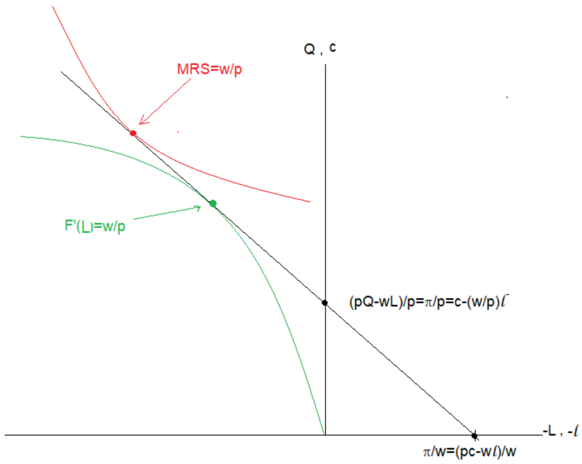

Figure 5: Robinson Crusoe Economy

5 Limitations of General Equilibrium

The standard model assumes:

- Perfect competition (price-taking behavior).
- No market frictions (e.g., transaction costs).
- Complete markets (prices for everything).
- Interactions only via prices (no externalities).

These are strong assumptions, making GE a theoretical benchmark rather than a perfect description of reality.

ECON0013 T2 Lec10 - Welfare Theorem & Public Goods

Ambrose W

1 Fundamental Theorems of Welfare Economics

1.1 First Fundamental Theorem of Welfare Economics

Statement: Any competitive equilibrium allocation (achieved through market trading) is Pareto efficient.

- This holds for both exchange economies and economies with production.
- It means that once a competitive equilibrium is reached, there's no way to make someone better off without making someone else worse off.
- Trade exhausts all possible Pareto improvements starting from the initial endowments.

Key Condition for Pareto Efficiency (Exchange Economy): Marginal Rates of Substitution (MRS) between any two goods must be equal for all consumers.

$$MRS_{1,2}^A = MRS_{1,2}^B = ... = \left(\frac{p_1^*}{p_2^*}\right)$$

Key Condition for Pareto Efficiency (Production Economy): MRS for consumers must equal the Marginal Rate of Transformation (MRT) in production.

$$MRS = MRT$$

For the Robinson Crusoe example ($c = F(L)$): $MRS_{\text{leisure},c} = F'(L)$.

1.2 Contract Curve

Definition: The set of all Pareto efficient allocations in an economy.

- In an Edgeworth box, it's the locus of points where indifference curves are tangent ($MRS^A = MRS^B$).
- Competitive equilibria lie on the contract curve.
- Allocations along the contract curve are Pareto-incomparable (moving along the curve helps one individual but hurts the other).

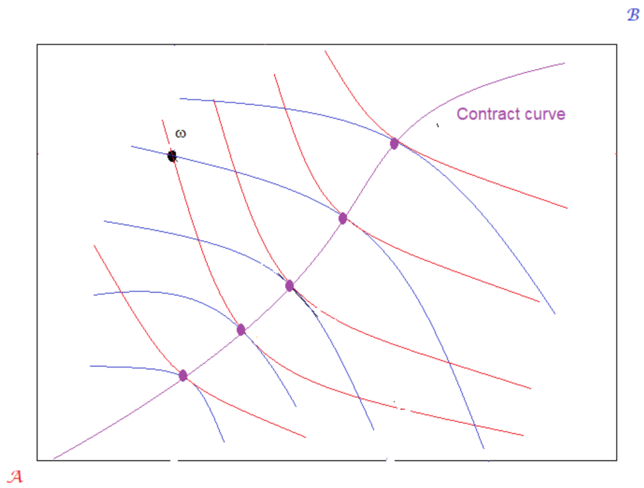

Figure 1: Contract Curve in Edgeworth Box

Example (Cobb-Douglas Utility): For $u^h(q_1^h, q_2^h) = (q_1^h)^{\alpha^h}(q_2^h)^{1-\alpha^h}$, Pareto efficiency requires:

$$\frac{\alpha^A}{1-\alpha^A}\frac{q_2^A}{q_1^A} = \frac{\alpha^B}{1-\alpha^B}\frac{q_2^B}{q_1^B}$$

If $\alpha^A = \alpha^B$, the contract curve is the diagonal of the Edgeworth box.

1.3 The Core

Definition: The subset of the contract curve consisting of Pareto efficient allocations that are also Pareto superior to the initial endowment ω.

- These are the allocations reachable through trade that make at least one person better off and no one worse off compared to their starting point.
- The competitive equilibrium allocation is always in the core (assuming trade occurs).

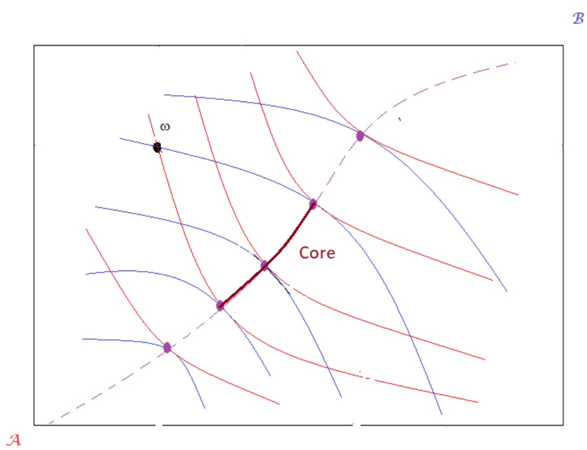

Figure 2: Core in Edgeworth Box

1.4 Second Fundamental Theorem of Welfare Economics

Statement: Any desired Pareto efficient allocation can be achieved (sustained) as a competitive equilibrium, provided there is an appropriate initial redistribution of endowments.

- Requires key assumptions: Convex preferences and convex production technologies.
- Implies that issues of efficiency and equity can be separated. Society can choose a desired efficient outcome (on the contract curve) based on equity considerations, and then achieve it through market mechanisms after lump-sum transfers of initial wealth/endowments.

Implication: The set of Pareto efficient allocations is essentially the same as the set of allocations achievable through competitive equilibrium with varying initial endowments.

2 Public Goods

2.1 Definition

Goods that are:

- **Non-rivalrous:** One person's consumption does not diminish the amount available for others.
- **Non-excludable:** It is difficult or impossible to prevent people from consuming the good once it is provided.

Examples: National defense, streetlights, clean air.

2.2 Efficient Provision of Public Goods

Consider one private good q and one public good Q. Let P be the constant Marginal Rate of Transformation (MRT) - the cost in terms of private goods to produce one unit of the public good. Utility for household h is $u^h(q^h, Q)$. The condition for Pareto efficient provision of the public good is the **Samuelson Condition**:

$$\sum_{h=1}^{H} MRS^h_{Q,q^h} = MRT(= P)$$

Where $MRS^h_{Q,q^h} = \frac{\partial u^h/\partial Q}{\partial u^h/\partial q^h}$ is household h's marginal willingness to pay (in terms of private good) for an additional unit of the public good. **Interpretation:** The sum of individual marginal benefits must equal the marginal cost of providing the public good.

Alternative Derivation (Production Economy): If output $Q = F(L)$ is a public good enjoyed by A and B, and input $L = l^A + l^B$ is private (e.g., labor/leisure), efficiency requires:

$$MRS^A_{\text{leisure},Q} + MRS^B_{\text{leisure},Q} = MRT_{L,Q}(= \frac{1}{F'(L)})$$

2.3 Private Provision and the Free-Rider Problem

If individuals can privately contribute to a public good (e.g., contribute amount G^h of private good endowment ω^h to fund Q, where $P \times G^h$ buys units of Q), they will likely under-provide it.

- Each individual h maximizes $u^h(\omega^h - PG^h, \sum_j G^j)$.
- In a Nash equilibrium where individuals choose their contribution G^h taking others' contributions as given, those who contribute positive amounts will do so only until their *individual* MRS equals the MRT:

$$MRS^h_{Q,q^h} = P$$

- This ignores the benefits their contribution provides to others (positive externality).
- Since $\sum MRS^h > MRS^h$ (for $H > 1$), the Samuelson condition $\sum MRS^h = P$ is not met.
- Result: Public goods are typically under-provided by the market due to the free-rider problem. This provides a rationale for government intervention (e.g., taxation and direct provision).

Example (Cobb-Douglas Utility): With $u^h(q^h, Q) = a \ln q^h + (1-a) \ln Q$, the efficient level of provision is $PQ = (1-a)\Omega$. Private provision (assuming N contributors) leads to $PQ = \frac{(1-a)\Omega}{1+(N-1)a}$, which is less than the efficient level for $N > 1$.

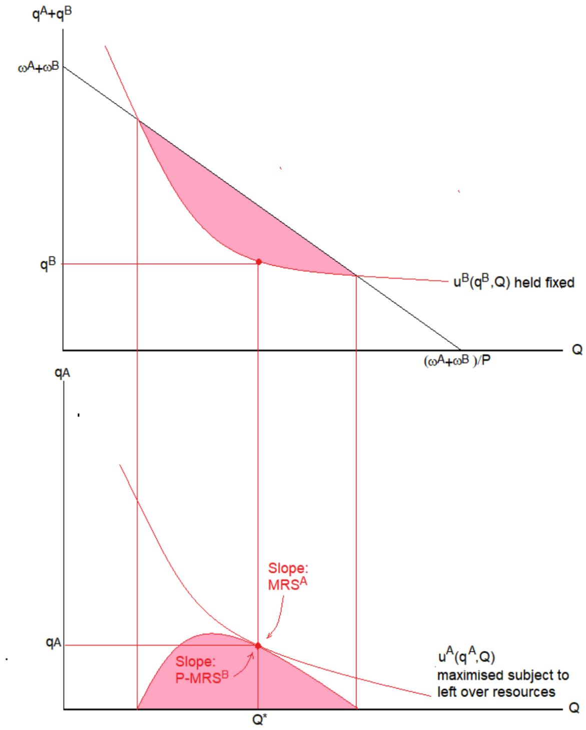

Figure 3: Efficient vs. Private Provision of Public Goods