Complete Worked Solutions

ECON0006 / ECON0010 Summer Exam 24/25

Ambrose Wang, April 2026

Contents

1 Question 1: matrices as vectors, tensor basis, and the Frobenius product

Concept box

A vector space is a set whose elements can be added and multiplied by real scalars while remaining inside the same set. The elements do not have to look like column vectors. Here the elements are 2×2 matrices. The tensor notation $e_i \otimes e_j$ is used concretely as the matrix unit $e_i e_j^\top$, namely the 2×2 matrix with a 1 in entry (i, j) and zeros elsewhere. The double-dot product is the Frobenius inner product:

$$A : B = \operatorname{tr}(AB^\top) = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} b_{ij}.$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

The standard basis of $\mathbb{R}^2$ is

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

1(a) Showing that the set of 2×2 matrices is a vector space

Let

$$\mathcal{M}_2 = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} : x_{ij} \in \mathbb{R} \right\}.$$

To show that this set is a real vector space, use the standard matrix operations.

First, addition is closed. If $A, B \in \mathcal{M}_2$, then

$$\begin{aligned} A + B &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}. \end{aligned}$$

Every entry is real, so $A + B \in \mathcal{M}_2$.

Second, scalar multiplication is closed. If $\alpha \in \mathbb{R}$, then

$$\alpha A = \alpha \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix} \in \mathcal{M}_2.$$

The remaining vector-space axioms, such as associativity, commutativity of addition, the existence of a zero element, additive inverses, and distributivity, follow entry by entry from the corresponding properties of real numbers. The zero vector in this vector space is the zero matrix

$$0_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and the additive inverse of A is

$$-A = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}.$$

The set of all 2×2 real matrices is a real vector space under ordinary matrix addition and scalar multiplication.

1(b) Linear independence of $E_1, \ldots, E_4$ and representation of A and B

Define the four matrices

$$E_1 = e_1 \otimes e_1 = e_1 e_1^\top, \quad E_2 = e_1 \otimes e_2 = e_1 e_2^\top, \quad E_3 = e_2 \otimes e_1 = e_2 e_1^\top, \quad E_4 = e_2 \otimes e_2 = e_2 e_2^\top.$$

Writing them explicitly,

$$E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$E_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

To prove linear independence, suppose

$$\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 = 0_{2\times 2}.$$

Substituting the matrices gives

$$\begin{aligned} \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 &= \alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}. \end{aligned}$$

For this to equal the zero matrix,

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

each entry must be zero. Therefore

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.$$

This proves linear independence.

The same calculation also shows spanning. Any 2×2 matrix can be written as a linear combination of these four matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} E_1 + a_{12} E_2 + a_{21} E_3 + a_{22} E_4,$$
$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = b_{11} E_1 + b_{12} E_2 + b_{21} E_3 + b_{22} E_4.$$

The matrices E_1, E_2, E_3, E_4 are linearly independent and span $\mathcal{M}_2$. Hence they form a basis of the vector space of 2×2 matrices. The coordinates of A in this basis are $(a_{11}, a_{12}, a_{21}, a_{22})$, and the coordinates of B are $(b_{11}, b_{12}, b_{21}, b_{22})$.

1(c) Orthonormality under the double-dot product and calculation of $A : B$

The double-dot product is defined by

$$A : B = \operatorname{tr}(AB^\top).$$

Since

$$AB^\top = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{11}b_{21} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{12} & a_{21}b_{21} + a_{22}b_{22} \end{pmatrix},$$

the trace is

$$\begin{aligned}A : B &= \operatorname{tr}(AB^\top)\\ &= (a_{11}b_{11} + a_{12}b_{12}) + (a_{21}b_{21} + a_{22}b_{22})\\ &= a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}.\end{aligned}$$

Now check orthonormality. The matrices $E_1, \ldots, E_4$ each have exactly one entry equal to 1 and all other entries equal to 0. Thus

$$E_i : E_i = 1$$

for each i. If $i \neq j$, then the 1s in E_i and E_j occur in different positions, so the sum of entry-by-entry products is zero:

$$E_i : E_j = 0 \quad (i \neq j).$$

Equivalently,

$$E_i : E_j = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

For example,

$$\begin{aligned}E_1 : E_2 &= \operatorname{tr}(E_1 E_2^\top)\\ &= \operatorname{tr}\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]\\ &= \operatorname{tr}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = 0,\end{aligned}$$

and

$$\begin{aligned}E_2 : E_2 &= \operatorname{tr}(E_2 E_2^\top)\\ &= \operatorname{tr}\left[\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]\\ &= \operatorname{tr}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = 1.\end{aligned}$$

The basis E_1, E_2, E_3, E_4 is orthonormal under $A : B = \operatorname{tr}(AB^\top)$, and

$$\boxed{A : B = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}}.$$

1(d) ECON0010 Only: components of $H = FG$

Let

$$F = \sum_{j=1}^{4} F_j E_j, \qquad G = \sum_{j=1}^{4} G_j E_j.$$

In matrix form, these are

$$F = \begin{pmatrix}F_1 & F_2\\ F_3 & F_4\end{pmatrix}, \qquad G = \begin{pmatrix}G_1 & G_2\\ G_3 & G_4\end{pmatrix}.$$

The matrix product $H = FG$ is

$$\begin{aligned}H &= \begin{pmatrix}F_1 & F_2\\ F_3 & F_4\end{pmatrix}\begin{pmatrix}G_1 & G_2\\ G_3 & G_4\end{pmatrix}\\ &= \begin{pmatrix}F_1G_1 + F_2G_3 & F_1G_2 + F_2G_4\\ F_3G_1 + F_4G_3 & F_3G_2 + F_4G_4\end{pmatrix}.\end{aligned}$$

Since $H = H_1E_1 + H_2E_2 + H_3E_3 + H_4E_4$, the entries of this matrix are the components (H_1, H_2, H_3, H_4).

$$\boxed{\begin{aligned} H_1 &= F_1G_1 + F_2G_3, & H_2 &= F_1G_2 + F_2G_4, \\ H_3 &= F_3G_1 + F_4G_3, & H_4 &= F_3G_2 + F_4G_4. \end{aligned}}$$

2 Question 2: constrained optimisation on a circle

Concept box

A Lagrange multiplier solves constrained optimisation by combining the objective and the constraint into one function. The constraint here is a circle: $(x_1 - 2)^2 + (x_2 - 2)^2 = 4$, centered at $(2, 2)$ with radius 2. At an interior regular constrained optimum, the gradient of the objective is parallel to the gradient of the constraint. The envelope theorem gives the derivative of the optimised value with respect to a parameter by differentiating the Lagrangian with respect to that parameter, holding the optimiser fixed.

The objective is

$$F(x) = x_1 x_2, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The parameter is

$$a = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

and the constraint is

$$g(x, a) = (x - a)^\top (x - a) - 4 = 0.$$

At $a = (2, 2)^\top$, this is

$$(x_1 - 2)^2 + (x_2 - 2)^2 - 4 = 0.$$

2(a) Lagrangian and first-order conditions

Use the Lagrangian

$$\mathcal{L}(x_1, x_2, \lambda) = x_1 x_2 - \lambda\big((x_1 - 2)^2 + (x_2 - 2)^2 - 4\big).$$

The first partial derivative with respect to λ is

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \lambda} &= -\big((x_1 - 2)^2 + (x_2 - 2)^2 - 4\big) \\ &= 4 - (x_1 - 2)^2 - (x_2 - 2)^2. \end{aligned}$$

The two partial derivatives with respect to x_1 and x_2 are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= x_2 - \lambda \cdot 2(x_1 - 2), \\ \frac{\partial \mathcal{L}}{\partial x_2} &= x_1 - \lambda \cdot 2(x_2 - 2). \end{aligned}$$

Setting all three derivatives equal to zero gives

$$4 - (x_1 - 2)^2 - (x_2 - 2)^2 = 0, \tag{1}$$
$$x_2 - 2\lambda(x_1 - 2) = 0, \tag{2}$$
$$x_1 - 2\lambda(x_2 - 2) = 0. \tag{3}$$

$$\boxed{\mathcal{L} = x_1 x_2 - \lambda\big((x_1 - 2)^2 + (x_2 - 2)^2 - 4\big)}$$

and the first-order conditions are (1)–(3).

2(b) Finding all local optima

Start from the two stationarity equations:

$$\begin{aligned} x_2 &= 2\lambda(x_1 - 2), \\ x_1 &= 2\lambda(x_2 - 2). \end{aligned}$$

Subtract the second equation from the first:

$$\begin{aligned} x_2 - x_1 &= 2\lambda(x_1 - 2) - 2\lambda(x_2 - 2) \\ &= 2\lambda(x_1 - x_2) \\ &= -2\lambda(x_2 - x_1). \end{aligned}$$

Hence

$$(1 + 2\lambda)(x_2 - x_1) = 0.$$

There are therefore two cases.

Case 1: $x_1 = x_2$. Let $x_1 = x_2 = z$. The constraint becomes

$$\begin{aligned} (z-2)^2 + (z-2)^2 &= 4, \\ 2(z-2)^2 &= 4, \\ (z-2)^2 &= 2, \\ z - 2 &= \pm\sqrt{2}. \end{aligned}$$

Thus

$$z = 2 + \sqrt{2} \quad \text{or} \quad z = 2 - \sqrt{2}.$$

For these candidates, use $x_2 = 2\lambda(x_1 - 2)$, or

$$z = 2\lambda(z - 2),$$

so

$$\lambda = \frac{z}{2(z-2)}.$$

If $z = 2 + \sqrt{2}$, then

$$\lambda = \frac{2+\sqrt{2}}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} + \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{1}{2}.$$

If $z = 2 - \sqrt{2}$, then

$$\lambda = \frac{2-\sqrt{2}}{-2\sqrt{2}} = -\frac{1}{\sqrt{2}} + \frac{1}{2}.$$

Case 2: $1 + 2\lambda = 0$. Then

$$\lambda = -\frac{1}{2}.$$

The first stationarity equation becomes

$$\begin{aligned} x_2 &= 2\left(-\frac{1}{2}\right)(x_1 - 2) \\ &= -(x_1 - 2) = 2 - x_1. \end{aligned}$$

Hence

$$x_1 + x_2 = 2.$$

Substitute $x_2 = 2 - x_1$ into the constraint:

$$\begin{aligned}
(x_1-2)^2 + (x_2-2)^2 &= 4,\\
(x_1-2)^2 + ((2-x_1)-2)^2 &= 4,\\
(x_1-2)^2 + (-x_1)^2 &= 4,\\
(x_1^2 - 4x_1 + 4) + x_1^2 &= 4,\\
2x_1^2 - 4x_1 &= 0,\\
2x_1(x_1-2) &= 0.
\end{aligned}$$

Thus $x_1 = 0$ or $x_1 = 2$. The corresponding values of $x_2 = 2 - x_1$ are $x_2 = 2$ and $x_2 = 0$.

The four constrained stationary points are

$$\boxed{\begin{aligned}
&(0,2), && \lambda = -\frac{1}{2},\\
&(2,0), && \lambda = -\frac{1}{2},\\
&(2-\sqrt{2}, 2-\sqrt{2}), && \lambda = \frac{1}{2} - \frac{1}{\sqrt{2}},\\
&(2+\sqrt{2}, 2+\sqrt{2}), && \lambda = \frac{1}{2} + \frac{1}{\sqrt{2}}.
\end{aligned}}$$

2(c) Nature of the local optima

The Hessian of the Lagrangian with respect to (x_1, x_2) is

$$\nabla^2_{xx}\mathcal{L} = \begin{pmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} -2\lambda & 1 \\ 1 & -2\lambda \end{pmatrix}.$$

The Lagrangian has been written as

$$\mathcal{L}(x,\lambda) = F(x) + \lambda h(x), \qquad h(x) = 4 - (x_1-2)^2 - (x_2-2)^2.$$

Hence the border in the bordered Hessian follows from

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda\, \partial x_1} = \frac{\partial h}{\partial x_1} = 4 - 2x_1, \qquad \frac{\partial^2 \mathcal{L}}{\partial \lambda\, \partial x_2} = \frac{\partial h}{\partial x_2} = 4 - 2x_2.$$

Equivalently, since $g(x) = (x_1-2)^2 + (x_2-2)^2 - 4 = -h(x)$, using ∇g instead of ∇h reverses the signs in the border only. The official convention for this Lagrangian is therefore

$$H_B = \begin{pmatrix} 0 & 4-2x_1 & 4-2x_2 \\ 4-2x_1 & -2\lambda & 1 \\ 4-2x_2 & 1 & -2\lambda \end{pmatrix}.$$

At the four stationary points, this gives

$$H_B(0,2) = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad H_B(2,0) = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 1 & 1 \\ 4 & 1 & 1 \end{pmatrix},$$

$$H_B(2+\sqrt{2}, 2+\sqrt{2}) = \begin{pmatrix} 0 & -2\sqrt{2} & -2\sqrt{2} \\ -2\sqrt{2} & -1-\sqrt{2} & 1 \\ -2\sqrt{2} & 1 & -1-\sqrt{2} \end{pmatrix}, \quad H_B(2-\sqrt{2}, 2-\sqrt{2}) = \begin{pmatrix} 0 & 2\sqrt{2} & 2\sqrt{2} \\ 2\sqrt{2} & \sqrt{2}-1 & 1 \\ 2\sqrt{2} & 1 & \sqrt{2}-1 \end{pmatrix}.$$

For one equality constraint in two variables, the bordered-Hessian test with the above convention is especially simple:

$$\boxed{\det(H_B) < 0 \Rightarrow \text{local minimum}, \qquad \det(H_B) > 0 \Rightarrow \text{local maximum.}}$$

Thus the classification can be made directly from the determinants of the four bordered Hessians.

Point $(0,2)$. The bordered Hessian is

$$H_B(0,2) = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Expanding along the first row gives

$$\begin{aligned} \det H_B(0,2) &= 0\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 4\begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} \\ &= -4(4\cdot 1 - 0\cdot 1) \\ &= -16 < 0. \end{aligned}$$

Therefore $(0,2)$ is a local minimum.

Point $(2,0)$. The bordered Hessian is

$$H_B(2,0) = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 1 & 1 \\ 4 & 1 & 1 \end{pmatrix}.$$

Expanding along the first row gives

$$\begin{aligned} \det H_B(2,0) &= 0\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} + 4\begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} \\ &= 4(0\cdot 1 - 4\cdot 1) \\ &= -16 < 0. \end{aligned}$$

Therefore $(2,0)$ is also a local minimum.

Point $(2-\sqrt{2}, 2-\sqrt{2})$. The bordered Hessian is

$$H_B(2-\sqrt{2}, 2-\sqrt{2}) = \begin{pmatrix} 0 & 2\sqrt{2} & 2\sqrt{2} \\ 2\sqrt{2} & \sqrt{2}-1 & 1 \\ 2\sqrt{2} & 1 & \sqrt{2}-1 \end{pmatrix}.$$

Expanding along the first row gives

$$\begin{aligned} \det H_B(2-\sqrt{2}, 2-\sqrt{2}) &= 0\begin{vmatrix} \sqrt{2}-1 & 1 \\ 1 & \sqrt{2}-1 \end{vmatrix} - 2\sqrt{2}\begin{vmatrix} 2\sqrt{2} & 1 \\ 2\sqrt{2} & \sqrt{2}-1 \end{vmatrix} + 2\sqrt{2}\begin{vmatrix} 2\sqrt{2} & \sqrt{2}-1 \\ 2\sqrt{2} & 1 \end{vmatrix} \\ &= -2\sqrt{2}[2\sqrt{2}(\sqrt{2}-1) - 2\sqrt{2}] + 2\sqrt{2}[2\sqrt{2} - 2\sqrt{2}(\sqrt{2}-1)] \\ &= -2\sqrt{2}[2\sqrt{2}(\sqrt{2}-2)] + 2\sqrt{2}[2\sqrt{2}(2-\sqrt{2})] \\ &= 8(2-\sqrt{2}) + 8(2-\sqrt{2}) \\ &= 32 - 16\sqrt{2} > 0. \end{aligned}$$

Therefore $(2-\sqrt{2}, 2-\sqrt{2})$ is a local maximum.

Point $(2+\sqrt{2}, 2+\sqrt{2})$**.** The bordered Hessian is

$$H_B(2+\sqrt{2}, 2+\sqrt{2}) = \begin{pmatrix} 0 & -2\sqrt{2} & -2\sqrt{2} \\ -2\sqrt{2} & -1-\sqrt{2} & 1 \\ -2\sqrt{2} & 1 & -1-\sqrt{2} \end{pmatrix}.$$

Expanding along the first row gives

$$\begin{aligned} \det H_B(2+\sqrt{2}, 2+\sqrt{2}) = 0 &\begin{vmatrix} -1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{vmatrix} - (-2\sqrt{2}) \begin{vmatrix} -2\sqrt{2} & 1 \\ -2\sqrt{2} & -1-\sqrt{2} \end{vmatrix} \\ &+ (-2\sqrt{2}) \begin{vmatrix} -2\sqrt{2} & -1-\sqrt{2} \\ -2\sqrt{2} & 1 \end{vmatrix} \\ &= 2\sqrt{2}\big[(-2\sqrt{2})(-1-\sqrt{2}) - (-2\sqrt{2})\big] - 2\sqrt{2}\big[(-2\sqrt{2}) - (-2\sqrt{2})(-1-\sqrt{2})\big] \\ &= 2\sqrt{2}\big[2\sqrt{2}(2+\sqrt{2})\big] - 2\sqrt{2}\big[-2\sqrt{2}(2+\sqrt{2})\big] \\ &= 8(2+\sqrt{2}) + 8(2+\sqrt{2}) \\ &= 32 + 16\sqrt{2} > 0. \end{aligned}$$

Therefore $(2+\sqrt{2}, 2+\sqrt{2})$ is a local maximum.

The objective values are

$$\begin{aligned} F(0,2) &= 0, \\ F(2,0) &= 0, \\ F(2-\sqrt{2}, 2-\sqrt{2}) &= (2-\sqrt{2})^2 = 6-4\sqrt{2}, \\ F(2+\sqrt{2}, 2+\sqrt{2}) &= (2+\sqrt{2})^2 = 6+4\sqrt{2}. \end{aligned}$$

Because the circle lies in the closed first quadrant and touches both axes, $x_1x_2 \geq 0$ on the feasible set. Hence $(0,2)$ and $(2,0)$ are global minima, and $(2+\sqrt{2}, 2+\sqrt{2})$ is the global maximum.

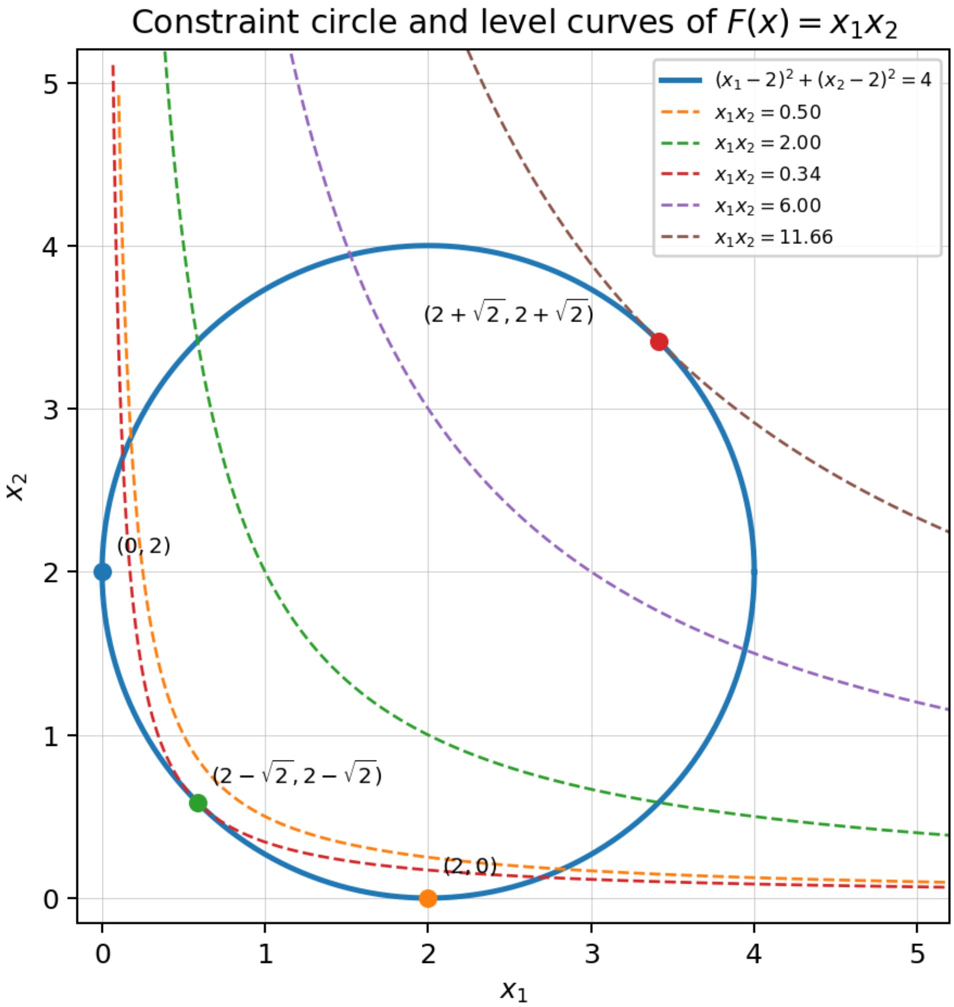

Figure 1: Constraint circle and level curves of $F(x) = x_1x_2$. This is included as a visual check; the local classification above uses only the bordered Hessian.

$$\boxed{(0,2) \text{ and } (2,0) \text{ are local, indeed global, minima.}}$$

$$\boxed{(2-\sqrt{2}, 2-\sqrt{2}) \text{ is a local maximum, and } (2+\sqrt{2}, 2+\sqrt{2}) \text{ is the global maximum.}}$$

2(d) ECON0010 Only: global optimum and the envelope theorem

From the objective values above, the global maximum is

$$x^*(a) = \begin{pmatrix} 2+\sqrt{2} \\ 2+\sqrt{2} \end{pmatrix} \quad \text{at} \quad a = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

with multiplier

$$\lambda^* = \frac{1}{2} + \frac{1}{\sqrt{2}}.$$

For a general parameter vector $a = (a_1, a_2)^\top$, write the constraint as

$$g(x, a) = (x_1 - a_1)^2 + (x_2 - a_2)^2 - 4 = 0.$$

The Lagrangian is

$$\mathcal{L}(x, a, \lambda) = F(x) - \lambda g(x, a).$$

Let
$$V(a) = F(x^*(a))$$
be the maximised value. The envelope theorem gives
$$\frac{\partial V}{\partial a_i} = \left.\frac{\partial \mathcal{L}}{\partial a_i}\right|_{x=x^*(a),\lambda=\lambda^*(a)}.$$
Since
$$\frac{\partial g}{\partial a_i} = -2(x_i - a_i),$$
it follows that
$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial a_i} &= -\lambda \frac{\partial g}{\partial a_i} \\ &= -\lambda[-2(x_i - a_i)] \\ &= 2\lambda(x_i - a_i).\end{aligned}$$
Therefore
$$\nabla_a V(a) = 2\lambda^*(a)(x^*(a) - a).$$
At $a = (2,2)^\top$,
$$x^*(a) - a = \begin{pmatrix}\sqrt{2}\\ \sqrt{2}\end{pmatrix}.$$
Thus
$$\begin{aligned}\nabla_a V(2,2) &= 2\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)\begin{pmatrix}\sqrt{2}\\ \sqrt{2}\end{pmatrix} \\ &= \left(1 + \frac{2}{\sqrt{2}}\right)\begin{pmatrix}\sqrt{2}\\ \sqrt{2}\end{pmatrix} \\ &= (1+\sqrt{2})\begin{pmatrix}\sqrt{2}\\ \sqrt{2}\end{pmatrix} \\ &= \begin{pmatrix}2+\sqrt{2}\\ 2+\sqrt{2}\end{pmatrix}.\end{aligned}$$

The global maximum is
$$\boxed{x^*(2,2) = \begin{pmatrix}2+\sqrt{2}\\ 2+\sqrt{2}\end{pmatrix}}.$$
The envelope-gradient of the optimised value is
$$\boxed{\left.\nabla_a F(x^*(a))\right|_{a=(2,2)^\top} = \begin{pmatrix}2+\sqrt{2}\\ 2+\sqrt{2}\end{pmatrix}.}$$
Strictly speaking, this is the gradient of the value function $V(a) = F(x^*(a))$, not the ordinary partial derivative of $F(x)$ with respect to a while holding x fixed.

3 Question 3: a two-dimensional linear difference equation

Concept box

A stationary solution of a difference equation is a value that remains unchanged over time. For a linear system $x_{t+1} = Ax_t + b$, deviations from the stationary state follow the homogeneous system $z_{t+1} = Az_t$. Eigenvalues determine long-run behaviour: if every eigenvalue has modulus less than 1, deviations vanish; if an eigenvalue has modulus greater than 1, generic deviations grow.

The system is

$$x_{t+1} - x_t = Mx_t - a, \qquad M = \begin{pmatrix} -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{3} \end{pmatrix}, \qquad a = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Equivalently,

$$x_{t+1} = (I + M)x_t - a.$$

Let

$$A_0 = I + M = \begin{pmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{3} \end{pmatrix}.$$

The notation A_0 is used here to avoid confusion with the matrix named A in Question 1.

3(a) Stationary solution

A stationary solution x_s satisfies

$$x_s - x_s = Mx_s - a.$$

The left-hand side is zero, so

$$Mx_s - a = 0, \qquad \text{or} \qquad Mx_s = a.$$

Thus

$$x_s = M^{-1}a.$$

Compute M^{-1}. For

$$M = \begin{pmatrix} -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{3} \end{pmatrix},$$

the determinant is

$$\begin{aligned} \det(M) &= \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{1}{9} - \frac{1}{4} \\ &= \frac{4}{36} - \frac{9}{36} \\ &= -\frac{5}{36}. \end{aligned}$$

Therefore

$$\begin{aligned} M^{-1} &= \frac{1}{\det(M)} \begin{pmatrix} -\frac{1}{3} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{3} \end{pmatrix} \\ &= -\frac{36}{5} \begin{pmatrix} -\frac{1}{3} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{12}{5} & \frac{18}{5} \\ \frac{18}{5} & \frac{12}{5} \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} x_s &= \begin{pmatrix} \frac{12}{5} & \frac{18}{5} \\ \frac{18}{5} & \frac{12}{5} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{12}{5} - \frac{18}{5} \\ \frac{18}{5} - \frac{12}{5} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{6}{5} \\ \frac{6}{5} \end{pmatrix}. \end{aligned}$$

$$\boxed{x_s = \begin{pmatrix} -\frac{6}{5} \\ \frac{6}{5} \end{pmatrix}.}$$

3(b) Asymptotic behaviour and stability

Let the deviation from the stationary state be

$$z_t = x_t - x_s.$$

Because x_s satisfies $x_s = A_0 x_s - a$, subtract the stationary equation from the original equation:

$$\begin{aligned} x_{t+1} - x_s &= (A_0 x_t - a) - (A_0 x_s - a) \\ &= A_0 (x_t - x_s). \end{aligned}$$

Hence

$$z_{t+1} = A_0 z_t.$$

The stability of x_s is therefore determined by the eigenvalues of

$$A_0 = \begin{pmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{3} \end{pmatrix}.$$

The characteristic equation is

$$\begin{aligned} 0 &= \det(A_0 - rI) \\ &= \det \begin{pmatrix} \frac{2}{3} - r & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{3} - r \end{pmatrix} \\ &= \left(\frac{2}{3} - r \right)^2 - \frac{1}{4}. \end{aligned}$$

Set this equal to zero:

$$\begin{aligned} \left(\frac{2}{3} - r \right)^2 &= \frac{1}{4}, \\ \frac{2}{3} - r &= \pm \frac{1}{2}. \end{aligned}$$

If $\frac{2}{3} - r = \frac{1}{2}$, then

$$r = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

If $\frac{2}{3} - r = -\frac{1}{2}$, then

$$r = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}.$$

The eigenvalues are therefore

$$r_1 = \frac{7}{6}, \qquad r_2 = \frac{1}{6}.$$

For asymptotic stability in a discrete-time system, all eigenvalues must satisfy $|r| < 1$. Since

$$\left|\frac{7}{6}\right| > 1,$$

the stationary state is not asymptotically stable for generic initial conditions.

The unstable eigenvalue $7/6$ corresponds to eigenvector $(1,1)^\top$. The stable eigenvalue $1/6$ corresponds to eigenvector $(1,-1)^\top$. Therefore the path converges only when the initial deviation has no component in the $(1,1)^\top$ direction. Equivalently,

$$(x_{1,0} - x_{s,1}) + (x_{2,0} - x_{s,2}) = 0.$$

Since $x_{s,1} + x_{s,2} = 0$, this condition is

$$x_{1,0} + x_{2,0} = 0.$$

The stationary state is not asymptotically stable because $A_0 = I + M$ has eigenvalue $7/6$, whose modulus exceeds 1. Convergence occurs only for initial conditions on the stable line

$$\boxed{x_{1,0} + x_{2,0} = 0.}$$

3(c) General solution

From the deviation equation $z_{t+1} = A_0 z_t$, repeated substitution gives

$$z_t = A_0^t z_0.$$

Since $z_t = x_t - x_s$, the general solution is

$$x_t = x_s + A_0^t (x_0 - x_s).$$

It remains to compute A_0^t.

For $r_1 = 7/6$, solve $(A_0 - r_1 I)v = 0$:

$$A_0 - \frac{7}{6}I = \begin{pmatrix} \frac{2}{3} - \frac{7}{6} & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{3} - \frac{7}{6} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

This gives $-v_1 + v_2 = 0$, so choose

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $r_2 = 1/6$,

$$A_0 - \frac{1}{6}I = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

which gives $v_1 + v_2 = 0$, so choose

$$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Let

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}.$$

Then

$$A_0 = PDP^{-1}.$$

The inverse of P is

$$P^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Therefore

$$A_0^t = PD^tP^{-1}.$$

Compute this product step by step:

$$\begin{aligned} PD^t &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \left(\frac{7}{6}\right)^t & 0 \\ 0 & \left(\frac{1}{6}\right)^t \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{7}{6}\right)^t & \left(\frac{1}{6}\right)^t \\ \left(\frac{7}{6}\right)^t & -\left(\frac{1}{6}\right)^t \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} A_0^t &= \begin{pmatrix} \left(\frac{7}{6}\right)^t & \left(\frac{1}{6}\right)^t \\ \left(\frac{7}{6}\right)^t & -\left(\frac{1}{6}\right)^t \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \left(\frac{7}{6}\right)^t + \left(\frac{1}{6}\right)^t & \left(\frac{7}{6}\right)^t - \left(\frac{1}{6}\right)^t \\ \left(\frac{7}{6}\right)^t - \left(\frac{1}{6}\right)^t & \left(\frac{7}{6}\right)^t + \left(\frac{1}{6}\right)^t \end{pmatrix}. \end{aligned}$$

Equivalently,

$$A_0^t = \frac{1}{2}6^{-t}\begin{pmatrix} 7^t + 1 & 7^t - 1 \\ 7^t - 1 & 7^t + 1 \end{pmatrix}.$$

$$\boxed{x_t = \begin{pmatrix} -\frac{6}{5} \\ \frac{6}{5} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \left(\frac{7}{6}\right)^t + \left(\frac{1}{6}\right)^t & \left(\frac{7}{6}\right)^t - \left(\frac{1}{6}\right)^t \\ \left(\frac{7}{6}\right)^t - \left(\frac{1}{6}\right)^t & \left(\frac{7}{6}\right)^t + \left(\frac{1}{6}\right)^t \end{pmatrix}\left(x_0 - \begin{pmatrix} -\frac{6}{5} \\ \frac{6}{5} \end{pmatrix}\right).}$$

3(d) ECON0010 Only: expression using the difference operator

Let the time series be

$$X = (\ldots, x_{t-1}, x_t, x_{t+1}, \ldots)^\top.$$

The forward difference operator Δ maps X to the time series whose t-th entry is

$$(\Delta X)_t = x_{t+1} - x_t.$$

In block-matrix form, Δ has $-I$ on the main block diagonal and I on the first upper block diagonal:

$$\Delta = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & -I & I & 0 & \\ & 0 & -I & I & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

The original equation is

$$x_{t+1} - x_t = Mx_t - a.$$

To write this as

$$\Delta X = \mathbb{M}X + \mathbb{A},$$

define the block-diagonal operator

$$\mathbb{M} = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & M & 0 & 0 & & \\ & 0 & M & 0 & & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

and the constant time series

$$\mathbb{A} = (\ldots, -a, -a, -a, \ldots)^\top.$$

Thus the t-th block of $\mathbb{M}X + \mathbb{A}$ is $Mx_t - a$, exactly matching the original equation.

Under the standard forward-difference convention $(\Delta X)_t = x_{t+1} - x_t$,

$$\boxed{\Delta X = \mathbb{M}X + \mathbb{A}, \qquad \mathbb{A} = (\ldots, -a, -a, -a, \ldots)^\top.}$$

If a course convention defines Δ with the opposite sign, both sides must be adjusted consistently.

4 Question 4: an opinion model, Jacobians, and Euler equations

Concept box

Each agent chooses an opinion σ_j. The first-order condition for each agent is one equation; collecting these equations gives a vector field. A vector field assigns a vector to each possible opinion vector. The Jacobian of the vector field records how those first-order conditions change when opinions change. Stability is governed by the eigenvalues of the Jacobian: in continuous time, strictly negative real parts imply local stability.

There are n agents. Agent j's opinion at time t is $\sigma_j(t)$, and the vector of opinions is

$$\sigma(t) = \begin{pmatrix} \sigma_1(t) \\ \vdots \\ \sigma_n(t) \end{pmatrix}.$$

The utility of agent j is

$$u_j(\sigma(t)) = \eta_j(t)\sigma_j(t) - \frac{R_j}{2}\sigma_j(t)^2 + \sigma_j(t)(\mathcal{C}\sigma(t))_j,$$

where $\mathcal{C}$ has zeros on the diagonal and $R_j > 0$. The matrix $\mathcal{C}$ is the interaction matrix. Define

$$\mathcal{R} = \operatorname{diag}(R_1, \dots, R_n).$$

4(a) First-order conditions and matrix form

For a fixed time t, suppress the time argument for clarity. Agent j's utility is

$$u_j(\sigma) = \eta_j\sigma_j - \frac{R_j}{2}\sigma_j^2 + \sigma_j(\mathcal{C}\sigma)_j.$$

The term $(\mathcal{C}\sigma)_j$ is

$$(\mathcal{C}\sigma)_j = \sum_{i=1}^{n} c_{ji}\sigma_i.$$

Since the diagonal of $\mathcal{C}$ is zero, $c_{jj} = 0$. Therefore $(\mathcal{C}\sigma)_j$ does not contain σ_j. Differentiating with respect to σ_j gives

$$\begin{aligned} \frac{\partial u_j}{\partial \sigma_j} &= \frac{\partial}{\partial \sigma_j}(\eta_j\sigma_j) - \frac{\partial}{\partial \sigma_j}\left(\frac{R_j}{2}\sigma_j^2\right) + \frac{\partial}{\partial \sigma_j}\left[\sigma_j(\mathcal{C}\sigma)_j\right] \\ &= \eta_j - R_j\sigma_j + (\mathcal{C}\sigma)_j. \end{aligned}$$

The first-order condition is

$$\eta_j - R_j\sigma_j + (\mathcal{C}\sigma)_j = 0.$$

Rearrange it:

$$R_j\sigma_j - (\mathcal{C}\sigma)_j = \eta_j.$$

Stacking these equations for $j = 1, \dots, n$ gives

$$(\mathcal{R} - \mathcal{C})\sigma^* = \eta.$$

If $\mathcal{R} - \mathcal{C}$ is invertible, then

$$\sigma^* = (\mathcal{R} - \mathcal{C})^{-1}\eta.$$

$$\boxed{(\mathcal{R} - \mathcal{C})\sigma^* = \eta, \qquad \sigma^* = (\mathcal{R} - \mathcal{C})^{-1}\eta \text{ if } \mathcal{R} - \mathcal{C} \text{ is invertible.}}$$

4(b) Jacobian of the first-order conditions and stability

Define the vector of first-order conditions as

$$v(\sigma) = \eta - \mathcal{R}\sigma + \mathcal{C}\sigma.$$

The equilibrium condition is

$$v(\sigma^*) = 0.$$

The Jacobian of this vector field is the matrix of partial derivatives

$$\mathcal{J} = \nabla_\sigma v(\sigma).$$

Because η is independent of σ,

$$\nabla_\sigma \eta = 0.$$

Also,

$$\nabla_\sigma(-\mathcal{R}\sigma) = -\mathcal{R},$$

and

$$\nabla_\sigma(\mathcal{C}\sigma) = \mathcal{C}.$$

Hence

$$\mathcal{J} = -\mathcal{R} + \mathcal{C}.$$

Entry by entry,

$$\mathcal{J}_{ji} = \frac{\partial v_j}{\partial \sigma_i} = \begin{cases} -R_j, & i = j, \\ c_{ji}, & i \neq j. \end{cases}$$

For gradient-learning dynamics in continuous time, a natural local adjustment equation is

$$\dot{\sigma} = \gamma v(\sigma), \qquad \gamma > 0.$$

Linearising around σ^*, let $\delta\sigma = \sigma - \sigma^*$. Since $v(\sigma^*) = 0$, the first-order approximation is

$$\dot{\delta\sigma} = \gamma \mathcal{J} \delta\sigma.$$

The factor $\gamma > 0$ does not change the signs of real parts of eigenvalues. Therefore local stability requires every eigenvalue of $\mathcal{J}$ to have strictly negative real part.

$$\boxed{\mathcal{J} = -\mathcal{R} + \mathcal{C}.}$$

The optimal opinion distribution is locally stable under continuous-time gradient learning if

$$\boxed{\mathrm{Re}(r_i(\mathcal{J})) < 0 \quad \text{for every eigenvalue } r_i(\mathcal{J}).}$$

4(c) Utility at the optimal opinion distribution

At the optimum, the first-order condition gives

$$\eta_j - R_j\sigma_j^* + (\mathcal{C}\sigma^*)_j = 0.$$

Thus

$$\eta_j + (\mathcal{C}\sigma^*)_j = R_j\sigma_j^*.$$

Now evaluate utility at σ^*:

$$\begin{aligned}
u_j(\sigma^*) &= \eta_j \sigma_j^* - \frac{R_j}{2}(\sigma_j^*)^2 + \sigma_j^*(\mathcal{C}\sigma^*)_j \\
&= \sigma_j^* \left[\eta_j + (\mathcal{C}\sigma^*)_j\right] - \frac{R_j}{2}(\sigma_j^*)^2 \\
&= \sigma_j^*(R_j\sigma_j^*) - \frac{R_j}{2}(\sigma_j^*)^2 \\
&= R_j(\sigma_j^*)^2 - \frac{R_j}{2}(\sigma_j^*)^2 \\
&= \frac{R_j}{2}(\sigma_j^*)^2.
\end{aligned}$$

From part (a),

$$\sigma^* = (\mathcal{R} - \mathcal{C})^{-1}\eta.$$

Since

$$\mathcal{J} = -\mathcal{R} + \mathcal{C} = -(\mathcal{R} - \mathcal{C}),$$

we have

$$\sigma^* = -(\mathcal{J}^{-1}\eta).$$

The minus sign disappears after squaring, so

$$(\sigma_j^*)^2 = \left[(\mathcal{J}^{-1}\eta)_j\right]^2.$$

$$\boxed{u_j(\sigma^*) = \frac{R_j}{2}(\sigma_j^*)^2 = \frac{R_j}{2}\left[((\mathcal{R} - \mathcal{C})^{-1}\eta)_j\right]^2 = \frac{R_j}{2}\left[(\mathcal{J}^{-1}\eta)_j\right]^2.}$$

The square is algebraically necessary: utility at the optimum is quadratic in the optimal opinion σ_j^*.

4(d) ECON0010 Only: Euler equations for the optimal path

Let the planned path be $x(t) = (x_1(t), \ldots, x_n(t))^\top$. The community starts from

$$x(0) = \sigma(0)$$

and reaches the optimal consensus state at time T:

$$x(T) = \sigma^*.$$

The utility-rate for agent j is

$$U_j(x(t), \dot{x}_j(t)) = u_j(x(t)) - \frac{\beta}{2}\dot{x}_j(t)^2, \qquad \beta > 0.$$

For each component $x_j(t)$, the Euler equation is

$$\frac{\partial U_j}{\partial x_j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U_j}{\partial \dot{x}_j}\right) = 0.$$

Compute the two terms. First,

$$\begin{aligned}
\frac{\partial U_j}{\partial x_j} &= \frac{\partial u_j}{\partial x_j} \\
&= \eta_j - R_j x_j + (\mathcal{C}x)_j.
\end{aligned}$$

Second,

$$\begin{aligned}\frac{\partial U_j}{\partial \dot{x}_j} &= \frac{\partial}{\partial \dot{x}_j}\left[-\frac{\beta}{2}\dot{x}_j^2\right] \\ &= -\beta \dot{x}_j.\end{aligned}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U_j}{\partial \dot{x}_j}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(-\beta \dot{x}_j) = -\beta \ddot{x}_j.$$

Substitute into the Euler equation:

$$\begin{aligned}\eta_j - R_j x_j + (\mathcal{C}x)_j - (-\beta \ddot{x}_j) = 0, \\ \eta_j - R_j x_j + (\mathcal{C}x)_j + \beta \ddot{x}_j = 0.\end{aligned}$$

Rearrange:

$$\beta \ddot{x}_j = R_j x_j - (\mathcal{C}x)_j - \eta_j.$$

In vector form,

$$\beta \ddot{x}(t) = (\mathcal{R} - \mathcal{C})x(t) - \eta.$$

Because $(\mathcal{R} - \mathcal{C})\sigma^* = \eta$, this can also be written as

$$\beta \ddot{x}(t) = (\mathcal{R} - \mathcal{C})(x(t) - \sigma^*).$$

The optimal path solves the boundary-value problem

$$\boxed{\beta \ddot{x}(t) = (\mathcal{R} - \mathcal{C})x(t) - \eta = (\mathcal{R} - \mathcal{C})(x(t) - \sigma^*)}$$

with boundary conditions

$$\boxed{x(0) = \sigma(0), \qquad x(T) = \sigma^*.}$$

Component by component,

$$\boxed{\beta \ddot{x}_j(t) = R_j x_j(t) - (\mathcal{C}x(t))_j - \eta_j(t).}$$

5 Question 5: nonlinear differential equations, stationary states, linearisation, and phase flow

Concept box

The model is a system of two first-order ordinary differential equations. A stationary state sets both time derivatives equal to zero. To study local behaviour around a stationary state, use the Jacobian matrix: it is the linear approximation to the nonlinear system. If both eigenvalues of the Jacobian have negative real parts, nearby paths converge back to the stationary state. Eigenvectors identify the principal directions along which the linearised flow moves.

The system is

$$\dot{k}(t) = sTk(t)^{1/2}h(t)^{1/2} - \delta k(t),$$
$$\dot{h}(t) = Hk(t)^{1/3}h(t)\left(1 - \frac{h(t)}{\chi}\right),$$

where

$$s, T, H, \delta, \chi > 0, \qquad k(t) \geq 0, \qquad h(t) \geq 0.$$

5(a) Type of differential equation

There are two state variables, $k(t)$ and $h(t)$, so this is a system. Only first derivatives, $\dot{k}(t)$ and $\dot{h}(t)$, appear, so it is first order. The independent variable is time t, and the system contains ordinary derivatives rather than partial derivatives, so it is ordinary. The right-hand sides contain powers such as $k^{1/2}h^{1/2}$, $k^{1/3}h$, and h^2, so the system is nonlinear.

$$\boxed{\text{The model is a system of two nonlinear first-order ordinary differential equations.}}$$

5(b) Stationary states

A stationary state (k, h) satisfies

$$0 = sTk^{1/2}h^{1/2} - \delta k, \tag{4}$$
$$0 = Hk^{1/3}h\left(1 - \frac{h}{\chi}\right). \tag{5}$$

Start with equation (4):

$$0 = sTk^{1/2}h^{1/2} - \delta k.$$

Factor out $k^{1/2}$:

$$0 = k^{1/2}\left(sTh^{1/2} - \delta k^{1/2}\right).$$

Therefore either

$$k = 0$$

or

$$sTh^{1/2} = \delta k^{1/2}.$$

If $k > 0$, square both sides:

$$s^2T^2h = \delta^2 k,$$
$$k = \frac{s^2T^2}{\delta^2}h.$$

Now use equation (5):

$$0 = Hk^{1/3}h\left(1 - \frac{h}{\chi}\right).$$

Since $H > 0$, the product is zero if

$$k = 0, \qquad h = 0, \qquad \text{or} \qquad 1 - \frac{h}{\chi} = 0.$$

The last condition gives

$$h = \chi.$$

Boundary stationary states. If $k = 0$, then both equations are zero for any $h \geq 0$. Therefore

$$(k, h) = (0, h), \qquad h \geq 0,$$

is a continuum of boundary stationary states. The special point $(0, 0)$ is included in this set.

Interior stationary state. For $k > 0$, equation (5) requires $h = \chi$, since $h = 0$ would imply $k = 0$ from equation (4). Substituting $h = \chi$ into

$$k = \frac{s^2T^2}{\delta^2}h$$

gives

$$k = \frac{s^2T^2}{\delta^2}\chi.$$

Therefore the economically meaningful stationary state is

$$(k^*, h^*) = \left(\frac{\chi s^2T^2}{\delta^2}, \chi\right).$$

All relevant stationary states are

$$\boxed{(k, h) = (0, h), \quad h \geq 0,}$$

and the positive economically meaningful stationary state is

$$\boxed{(k^*, h^*) = \left(\frac{\chi s^2T^2}{\delta^2}, \chi\right).}$$

5(c) Linear approximation around the meaningful stationary state and stability

Define

$$f_1(k, h) = sTk^{1/2}h^{1/2} - \delta k,$$
$$f_2(k, h) = Hk^{1/3}h\left(1 - \frac{h}{\chi}\right).$$

Then

$$\begin{pmatrix} \dot{k} \\ \dot{h} \end{pmatrix} = \begin{pmatrix} f_1(k, h) \\ f_2(k, h) \end{pmatrix}.$$

The Jacobian is

$$J(k, h) = \begin{pmatrix} \frac{\partial f_1}{\partial k} & \frac{\partial f_1}{\partial h} \\ \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial h} \end{pmatrix}.$$

Compute each partial derivative.

For f_1,

$$\frac{\partial f_1}{\partial k} = sT \cdot \frac{1}{2} k^{-1/2} h^{1/2} - \delta,$$
$$\frac{\partial f_1}{\partial h} = sT \cdot \frac{1}{2} k^{1/2} h^{-1/2}.$$

For f_2, write

$$f_2(k,h) = Hk^{1/3}\left(h - \frac{h^2}{\chi}\right).$$

Then

$$\frac{\partial f_2}{\partial k} = H \cdot \frac{1}{3} k^{-2/3}\left(h - \frac{h^2}{\chi}\right),$$
$$\frac{\partial f_2}{\partial h} = Hk^{1/3}\left(1 - \frac{2h}{\chi}\right).$$

Now evaluate at

$$k^* = \frac{\chi s^2 T^2}{\delta^2}, \qquad h^* = \chi.$$

First,

$$\begin{aligned}
\left.\frac{\partial f_1}{\partial k}\right|_{(k^*,h^*)} &= \frac{sT}{2}\left(\frac{h^*}{k^*}\right)^{1/2} - \delta \\
&= \frac{sT}{2}\left(\frac{\chi}{\chi s^2 T^2/\delta^2}\right)^{1/2} - \delta \\
&= \frac{sT}{2}\left(\frac{\delta^2}{s^2 T^2}\right)^{1/2} - \delta \\
&= \frac{sT}{2} \cdot \frac{\delta}{sT} - \delta \\
&= \frac{\delta}{2} - \delta \\
&= -\frac{\delta}{2}.
\end{aligned}$$

Second,

$$\begin{aligned}
\left.\frac{\partial f_1}{\partial h}\right|_{(k^*,h^*)} &= \frac{sT}{2}\left(\frac{k^*}{h^*}\right)^{1/2} \\
&= \frac{sT}{2}\left(\frac{\chi s^2 T^2/\delta^2}{\chi}\right)^{1/2} \\
&= \frac{sT}{2} \cdot \frac{sT}{\delta} \\
&= \frac{s^2 T^2}{2\delta}.
\end{aligned}$$

Third,

$$\begin{aligned}
\left.\frac{\partial f_2}{\partial k}\right|_{(k^*,h^*)} &= \frac{H}{3}(k^*)^{-2/3}\left(\chi - \frac{\chi^2}{\chi}\right) \\
&= \frac{H}{3}(k^*)^{-2/3}(\chi - \chi) \\
&= 0.
\end{aligned}$$

Fourth,

$$\left.\frac{\partial f_2}{\partial h}\right|_{(k^*,h^*)} = H(k^*)^{1/3}\left(1-\frac{2\chi}{\chi}\right)$$
$$= H(k^*)^{1/3}(1-2)$$
$$= -H(k^*)^{1/3}$$
$$= -H\left(\frac{\chi s^2 T^2}{\delta^2}\right)^{1/3}.$$

Thus the Jacobian at the meaningful stationary state is

$$J = \begin{pmatrix} -\frac{\delta}{2} & \frac{s^2T^2}{2\delta} \\ 0 & -H\left(\frac{\chi s^2T^2}{\delta^2}\right)^{1/3} \end{pmatrix}.$$

For small deviations

$$\delta k(t) = k(t) - k^*, \qquad \delta h(t) = h(t) - h^*,$$

the linear approximation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\delta k(t)\\ \delta h(t)\end{pmatrix} = J\begin{pmatrix}\delta k(t)\\ \delta h(t)\end{pmatrix}.$$

Because J is triangular, its eigenvalues are its diagonal entries:

$$r_1 = -\frac{\delta}{2}, \qquad r_2 = -H\left(\frac{\chi s^2T^2}{\delta^2}\right)^{1/3}.$$

Since all parameters are positive,

$$r_1 < 0, \qquad r_2 < 0.$$

Therefore the meaningful stationary state is locally asymptotically stable.

$$\boxed{J = \begin{pmatrix} -\frac{\delta}{2} & \frac{s^2T^2}{2\delta} \\ 0 & -H\left(\frac{\chi s^2T^2}{\delta^2}\right)^{1/3} \end{pmatrix}.}$$

The linearised system is

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\delta k(t)\\ \delta h(t)\end{pmatrix} = J\begin{pmatrix}\delta k(t)\\ \delta h(t)\end{pmatrix}.}$$

Both eigenvalues are strictly negative, so the meaningful stationary state is locally asymptotically stable.

5(d) ECON0010 Only: eigenvectors and flow around the stationary state

For compact notation, define

$$K^* = \frac{\chi s^2T^2}{\delta^2}, \qquad \alpha = H(K^*)^{1/3}, \qquad c = \frac{s^2T^2}{2\delta}.$$

Then the Jacobian is

$$J = \begin{pmatrix} -\frac{\delta}{2} & c \\ 0 & -\alpha \end{pmatrix}.$$

The eigenvalues are

$$r_1 = -\frac{\delta}{2}, \qquad r_2 = -\alpha.$$

Eigenvector for $r_1 = -\delta/2$. Solve

$$(J - r_1 I)v = 0.$$

Since $r_1 = -\delta/2$,

$$J - r_1 I = \begin{pmatrix} 0 & c \\ 0 & -\alpha + \frac{\delta}{2} \end{pmatrix}.$$

If the eigenvalues are distinct, then $-\alpha + \delta/2 \neq 0$. The equations imply

$$cv_2 = 0,$$

so $v_2 = 0$. Choose $v_1 = 1$. Hence

$$v^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvector for $r_2 = -\alpha$. Solve

$$(J - r_2 I)v = 0.$$

Since $r_2 = -\alpha$,

$$J - r_2 I = \begin{pmatrix} \alpha - \frac{\delta}{2} & c \\ 0 & 0 \end{pmatrix}.$$

The equation is

$$\left(\alpha - \frac{\delta}{2}\right) v_1 + cv_2 = 0.$$

Set $v_2 = 1$. Then

$$\begin{aligned} v_1 &= -\frac{c}{\alpha - \delta/2} \\ &= \frac{c}{\delta/2 - \alpha} \\ &= \frac{\frac{s^2T^2}{2\delta}}{\frac{\delta}{2} - H(K^*)^{1/3}} \\ &= \frac{s^2T^2}{\delta\left(\delta - 2H(K^*)^{1/3}\right)}. \end{aligned}$$

Therefore a right eigenvector is

$$v^{(2)} = \begin{pmatrix} \frac{s^2T^2}{\delta\left(\delta - 2H\left(\frac{\chi s^2 T^2}{\delta^2}\right)^{1/3}\right)} \\ 1 \end{pmatrix}.$$

For completeness, the left eigenvectors are eigenvectors of $J^\top$. For the distinct-eigenvalue case, they can be chosen as

$$w^{(1)} = \begin{pmatrix} -\frac{\delta^2 - 2\delta H(K^*)^{1/3}}{s^2T^2} \\ 1 \end{pmatrix}, \qquad w^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The right eigenvectors are more important for drawing the phase flow, because they give directions of motion in the $(\delta k, \delta h)$ plane.

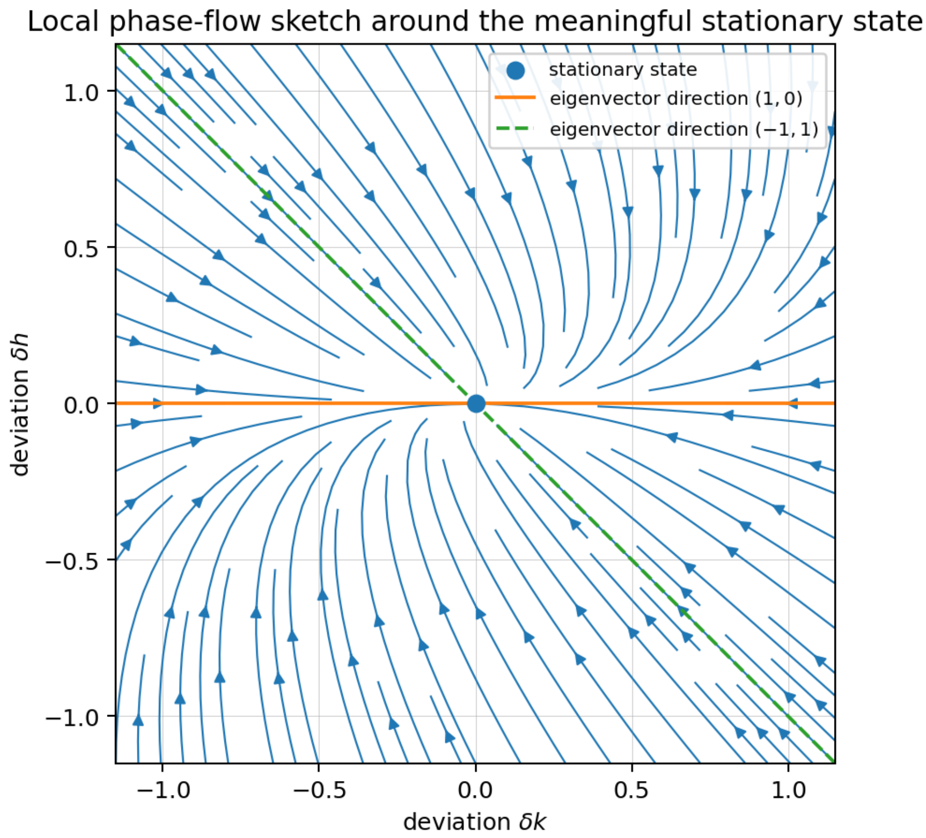

Figure 2: Local phase-flow sketch in deviations from the meaningful stationary state. The plotted representative parameters are $\delta = s = T = H = \chi = 1$, giving the stable Jacobian $J = \begin{pmatrix} -1/2 & 1/2 \\ 0 & -1 \end{pmatrix}$. The qualitative feature of interest is convergence toward the origin and non-orthogonal eigendirections.

The phase diagram is drawn in the deviation variables $(\delta k, \delta h)$. The stationary state itself is therefore at the origin of the diagram. Since both eigenvalues are negative, arrows point toward the origin. The eigendirections show the straight-line directions along which the linearised system moves without changing direction. Because J is generally not symmetric, the right eigenvectors are generally not perpendicular; the flow can therefore approach the stationary state along skewed directions rather than orthogonal axes.

If $\alpha = \delta/2$, the two eigenvalues coincide. The eigenvalue is still negative, so the state remains locally stable, but the matrix may have only one independent right eigenvector because the off-diagonal term $c > 0$. This is a degenerate repeated-root case; the generic distinct-eigenvalue formula above applies whenever $\alpha \neq \delta/2$.

For the generic distinct-eigenvalue case,

$$\boxed{v^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v^{(2)} = \begin{pmatrix} \dfrac{s^2T^2}{\delta\left(\delta - 2H\left(\frac{\chi s^2 T^2}{\delta^2}\right)^{1/3}\right)} \\ 1 \end{pmatrix}.}$$

Both eigenvalues are negative, so nearby paths converge to the meaningful stationary state.