ECON0010 — Full Worked Solutions

Problem Set 2, 2026 & LSA, 2025

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Contents

1 Notation glossary

The original papers use a mixture of ordinary mathematical notation and Mathematica-style notation. Throughout this manual I use the following conventions.

Original or informal notation	Standard notation used here
`f[x]`	$f(x)$
`Log[x]`	$\ln x$
`{x,y}`	$\begin{pmatrix} x \\ y \end{pmatrix}$, unless row-vector notation is explicitly intended
`A . x`	$A\mathbf{x}$
`x . y`	$\mathbf{x}^\top\mathbf{y}$, where appropriate
Lower-case bold letters	Vectors, e.g. $\mathbf{x}, \mathbf{v}, \mathbf{e}_1$
Upper-case letters	Matrices, e.g. A, J, H, M
I_n	The $n \times n$ identity matrix
$\mathbf{u} \otimes \mathbf{v}$	The outer product $\mathbf{u}\mathbf{v}^\top$, unless a different interpretation is forced by context
$(\boldsymbol{\sigma} - \mathbf{1})^2$	The Euclidean squared norm $(\boldsymbol{\sigma} - \mathbf{1})^\top(\boldsymbol{\sigma} - \mathbf{1})$, where $\mathbf{1} = (1, 1)^\top$
`x'[t]` and `x"[t]`	$\dot{x}(t)$ and $\ddot{x}(t)$
`x[t+1]-x[t]`	$\mathbf{x}_{t+1} - \mathbf{x}_t$
Eigenvalues	Usually denoted by λ_i, m_i, or μ_i to avoid conflict with model parameters
Reciprocal basis	Vectors $\mathbf{b}_j$ satisfying $\mathbf{b}_j^\top\mathbf{a}_k = \delta_{jk}$, where $\delta_{jk} = 1$ if $j = k$ and 0 otherwise

2 Solutions to ECON0010 Problem Set 2, 2026

2.1 Question 1: Matrix algebra

1(a)

Question in standard notation

Given $\mathbf{e}_1 = (1, 0, 0)^\top$, $\mathbf{e}_2 = (0, 1, 0)^\top$, and $\mathbf{e}_3 = (0, 0, 1)^\top$, calculate

$$A = \mathbf{e}_1 \otimes \mathbf{e}_1 + a\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_2 + b\mathbf{e}_3 \otimes \mathbf{e}_3 + c\mathbf{e}_3 \otimes \mathbf{e}_2.$$

Since $\mathbf{u} \otimes \mathbf{v} = \mathbf{u}\mathbf{v}^\top$, each term places its coefficient in a particular matrix entry:

$$\mathbf{e}_1\mathbf{e}_1^\top = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a\mathbf{e}_1\mathbf{e}_2^\top = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$-\mathbf{e}_2\mathbf{e}_2^\top = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b\mathbf{e}_3\mathbf{e}_3^\top = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{pmatrix}, \quad c\mathbf{e}_3\mathbf{e}_2^\top = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix}.$$

Adding gives

Final answer

$$\boxed{A = \begin{pmatrix} 1 & a & 0 \\ 0 & -1 & 0 \\ 0 & c & b \end{pmatrix}.}$$

1(b)

Question in standard notation

Solve the eigenvalue problem $A\mathbf{a}_i = \lambda_i \mathbf{a}_i$ for $i = 1, 2, 3$. Find the characteristic equation and the eigenvalues.

The characteristic equation is $\det(A - \lambda I_3) = 0$. Here

$$A - \lambda I_3 = \begin{pmatrix} 1-\lambda & a & 0 \\ 0 & -1-\lambda & 0 \\ 0 & c & b-\lambda \end{pmatrix}.$$

Expanding down the first column gives

$$\det(A - \lambda I_3) = (1-\lambda)(-1-\lambda)(b-\lambda).$$

Thus the characteristic equation is

$$(1-\lambda)(-1-\lambda)(b-\lambda) = 0,$$

so the eigenvalues are 1, -1, and b.

Final answer

$$\boxed{(1-\lambda)(-1-\lambda)(b-\lambda) = 0, \qquad \lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = b.}$$

1(c)

Question in standard notation

Find eigenvectors for the three eigenvalues and decide whether the three eigenvectors are orthogonal.

For $\lambda_1 = 1$, solving $(A - I)\mathbf{v} = 0$ gives $v_2 = 0$ and, generically, $v_3 = 0$, so we may take

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = -1$, solving $(A + I)\mathbf{v} = 0$ gives

$$2v_1 + av_2 = 0, \qquad cv_2 + (b+1)v_3 = 0.$$

Taking $v_2 = 1$ gives

$$\mathbf{a}_2 = \begin{pmatrix} -a/2 \\ 1 \\ -c/(b+1) \end{pmatrix}, \qquad b \neq -1.$$

For $\lambda_3 = b$, the generic eigenvector is

$$\mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad b \neq 1, -1.$$

Orthogonality would require all dot products to be zero. But

$$\mathbf{a}_1^\top \mathbf{a}_2 = -\frac{a}{2}, \qquad \mathbf{a}_2^\top \mathbf{a}_3 = -\frac{c}{b+1}.$$

These are not zero in general.

Final answer

$$\boxed{\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} -a/2 \\ 1 \\ -c/(b+1) \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.}$$

$\boxed{\text{The eigenvectors are not orthogonal in general.}}$

1(d)

Question in standard notation

Find vectors $\mathbf{b}_j$ forming a reciprocal basis to the eigenvectors from part (c), so that $\mathbf{b}_j^\top \mathbf{a}_k = \delta_{jk}$.

Place the eigenvectors in a matrix P:

$$P = \begin{pmatrix} 1 & -a/2 & 0 \\ 0 & 1 & 0 \\ 0 & -c/(b+1) & 1 \end{pmatrix}.$$

The rows of P^{-1} give $\mathbf{b}_j^\top$. Direct inversion gives

$$P^{-1} = \begin{pmatrix} 1 & a/2 & 0 \\ 0 & 1 & 0 \\ 0 & c/(b+1) & 1 \end{pmatrix}.$$

Therefore

Final answer

$$\boxed{\mathbf{b}_1 = \begin{pmatrix} 1 \\ a/2 \\ 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} 0 \\ c/(b+1) \\ 1 \end{pmatrix}.}$$

For example,

$$\mathbf{b}_1^\top \mathbf{a}_2 = -\frac{a}{2} + \frac{a}{2} = 0, \qquad \mathbf{b}_3^\top \mathbf{a}_2 = \frac{c}{b+1} - \frac{c}{b+1} = 0.$$

2.2 Question 2: Optimization

2(a)

Question in standard notation

Let

$$f(x,y) = \ln(xy^b) - \beta(x^2+y^2), \qquad 0<b<1, \quad \beta>0.$$

Maximize f over $x \geq 0$, $y \geq 0$. Find the first-order conditions.

Because $\ln(xy^b) = \ln x + b\ln y$, the logarithm requires $x>0$ and $y>0$. Thus any relevant optimum is interior. The first partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{1}{x} - 2\beta x, \qquad \frac{\partial f}{\partial y} = \frac{b}{y} - 2\beta y.$$

Final answer

$$\boxed{\frac{1}{x} - 2\beta x = 0, \qquad \frac{b}{y} - 2\beta y = 0.}$$

2(b)

Question in standard notation

Solve the first-order conditions and explain why only one solution is relevant.

The first equation gives $x^2 = 1/(2\beta)$, so formally

$$x = \pm\frac{1}{\sqrt{2\beta}}.$$

The second gives $y^2 = b/(2\beta)$, so formally

$$y = \pm\sqrt{\frac{b}{2\beta}}.$$

There are four algebraic sign combinations. However, the logarithm and the constraint $\mathbf{x} \geq \mathbf{0}$ require $x>0$ and $y>0$. Therefore only the positive solution is economically and mathematically relevant.

Final answer

$$\boxed{\text{Algebraic solutions: } 4, \qquad x^* = \frac{1}{\sqrt{2\beta}}, \quad y^* = \sqrt{\frac{b}{2\beta}}.}$$

2(c)

Question in standard notation

Calculate the Hessian in general and at the relevant local optimum. Classify the optimum.

The Hessian is diagonal:

$$H(x,y) = \begin{pmatrix} -1/x^2 - 2\beta & 0 \\ 0 & -b/y^2 - 2\beta \end{pmatrix}.$$

At $x^* = 1/\sqrt{2\beta}$ and $y^* = \sqrt{b/(2\beta)}$,

$$-\frac{1}{(x^*)^2} - 2\beta = -2\beta - 2\beta = -4\beta,$$

and

$$-\frac{b}{(y^*)^2} - 2\beta = -2\beta - 2\beta = -4\beta.$$

Since $\beta > 0$, $H^* = -4\beta I_2$ is negative definite.

Final answer

$$\boxed{H(x,y) = \begin{pmatrix} -1/x^2 - 2\beta & 0 \\ 0 & -b/y^2 - 2\beta \end{pmatrix}, \qquad H^* = -4\beta I_2.}$$

$$\boxed{(x^*, y^*) \text{ is a strict local maximum.}}$$

2(d)

Question in standard notation

Use the envelope theorem to find $\partial f(x^*, y^*)/\partial b$.

Holding x and y fixed,

$$\frac{\partial f}{\partial b} = \ln y.$$

By the envelope theorem we evaluate this at the optimum:

$$\frac{\partial}{\partial b} f(x^*(b), y^*(b), b) = \ln y^* = \ln \sqrt{\frac{b}{2\beta}}.$$

Final answer

$$\boxed{\frac{\partial}{\partial b} f(x^*, y^*) = \frac{1}{2} \ln\left(\frac{b}{2\beta}\right).}$$

2.3 Question 3: Difference equations

3(a)

Question in standard notation

Let

$$\mathbf{v}(x,y) = \begin{pmatrix} y + x - 2x^2 \\ y + xy - 2y^2 \end{pmatrix}$$

and consider $\mathbf{x}_{t+1} - \mathbf{x}_t = \mathbf{v}(\mathbf{x}_t)$. Check whether $\mathbf{v}$ is a gradient field and find all stationary states.

For a two-dimensional vector field $\mathbf{v} = (v_1, v_2)$ to be a gradient field on a simply connected region, we need

$$\frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial x}.$$

Here

$$\frac{\partial v_1}{\partial y} = 1, \qquad \frac{\partial v_2}{\partial x} = y.$$

These are not equal for all (x, y), so the field is not a gradient field.

Stationary states solve $v_1 = v_2 = 0$:

$$y + x - 2x^2 = 0, \qquad y(1 + x - 2y) = 0.$$

If $y = 0$, then $x - 2x^2 = 0$, so $x = 0$ or $x = 1/2$. If $1 + x - 2y = 0$, then $y = (1 + x)/2$. Substituting into the first equation gives

$$\frac{1+x}{2} + x - 2x^2 = 0 \quad \Longleftrightarrow \quad 4x^2 - 3x - 1 = 0,$$

so $x = 1$ or $x = -1/4$. The corresponding y values are 1 and 3/8.

Final answer

$$\boxed{\mathbf{v} \text{ is not a gradient field.}}$$

$$\boxed{(0,0), \quad (1/2,0), \quad (1,1), \quad (-1/4,3/8).}$$

3(b)

Question in standard notation

Calculate the Jacobian of the vector field at arbitrary (x, y).

The Jacobian collects all first partial derivatives:

$$J(x,y) = \begin{pmatrix} \partial v_1/\partial x & \partial v_1/\partial y \\ \partial v_2/\partial x & \partial v_2/\partial y \end{pmatrix} = \begin{pmatrix} 1-4x & 1 \\ y & 1+x-4y \end{pmatrix}.$$

Final answer

$$\boxed{J(x,y) = \begin{pmatrix} 1-4x & 1 \\ y & 1+x-4y \end{pmatrix}.}$$

3(c)

Question in standard notation

Determine how many stationary states are strictly stable.

The nonlinear difference equation is

$$\mathbf{x}_{t+1} = \mathbf{x}_t + \mathbf{v}(\mathbf{x}_t).$$

Near a stationary state $\mathbf{x}_s$, small deviations satisfy

$$\delta_{t+1} = (I_2 + J(\mathbf{x}_s))\delta_t.$$

Strict stability in discrete time requires all eigenvalues of $I_2 + J(\mathbf{x}_s)$ to have modulus less than 1.

Stationary state	Eigenvalues of $I+J$	Strictly stable?
$(0,0)$	$2,2$	no
$(1/2,0)$	$0,5/2$	no
$(1,1)$	$\frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$	no
$(-1/4,3/8)$	$\frac{13\pm\sqrt{145}}{8}$	no

At least one eigenvalue has modulus greater than 1 in every case.

Final answer

Number of strictly stable stationary states: 0.

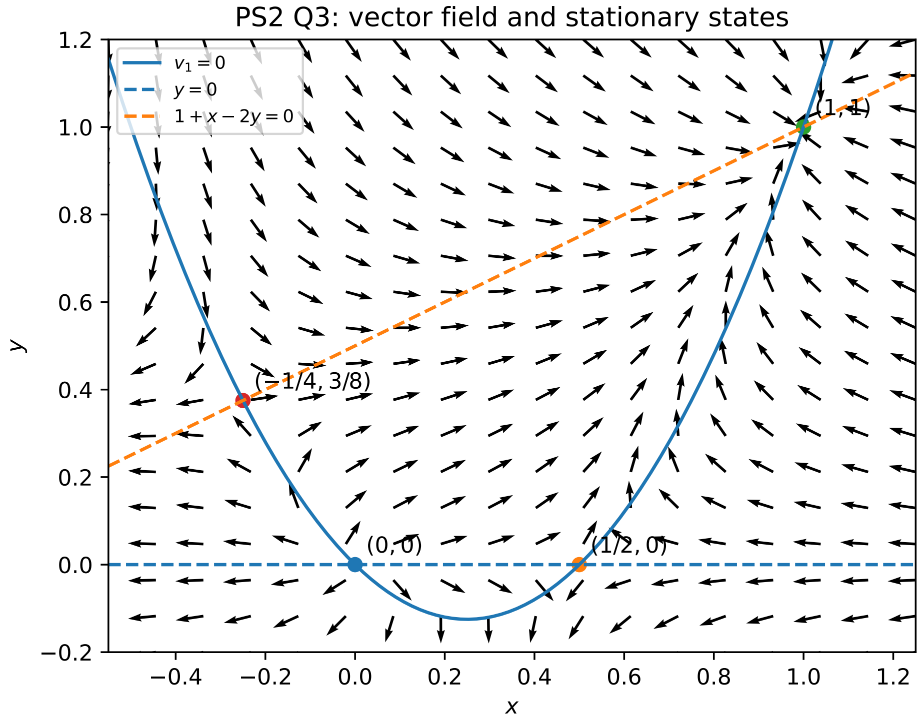

Figure 1: Vector field, nullclines, and stationary states for PS2 Question 3. The arrows are normalized to show direction clearly.

3(d)

Question in standard notation

Choose one stationary state with $J^\top \neq J$, compute the relevant eigenvectors, and comment on how small deviations evolve as $t \to \infty$.

The Jacobian is symmetric only when $y = 1$. Among the four stationary states, only $(1,1)$ has $y = 1$. Therefore there are three stationary states with non-symmetric Jacobian. Choose $(1/2, 0)$.

At $(1/2, 0)$,

$$J = \begin{pmatrix} -1 & 1 \\ 0 & 3/2 \end{pmatrix}, \qquad B := I + J = \begin{pmatrix} 0 & 1 \\ 0 & 5/2 \end{pmatrix}.$$

The eigenvectors of B are

$$\mu_1 = 0, \quad \mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad \mu_2 = \frac{5}{2}, \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ 5/2 \end{pmatrix}.$$

Thus a deviation component in the $\mathbf{r}_1$ direction disappears after one period, while any component in the $\mathbf{r}_2$ direction grows like $(5/2)^t$.

Final answer

$$\boxed{\text{There are 3 stationary states with } J^\top \neq J.}$$

$$\boxed{J(1/2, 0) = \begin{pmatrix} -1 & 1 \\ 0 & 3/2 \end{pmatrix}, \quad \mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ 5/2 \end{pmatrix}.}$$

$$\boxed{\text{Generic deviations diverge because } |5/2| > 1.}$$

2.4 Question 4: Constrained optimization

4(a)

Question in standard notation

An agent chooses $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)^\top > \mathbf{0}$. Utility is

$$u(\boldsymbol{\sigma}) = \sqrt{\sigma_1 \sigma_2},$$

and the constraint is

$$g(\boldsymbol{\sigma}) = (\boldsymbol{\sigma} - \mathbf{1})^2 - 1 = (\sigma_1 - 1)^2 + (\sigma_2 - 1)^2 - 1 = 0.$$

Give the Lagrangian, sketch isoquants and the constraint, and comment on expected local constrained optima.

Using the sign convention $\mathcal{L} = u - \lambda g$,

$$\mathcal{L}(\sigma_1, \sigma_2, \lambda) = \sqrt{\sigma_1 \sigma_2} - \lambda \left((\sigma_1 - 1)^2 + (\sigma_2 - 1)^2 - 1 \right).$$

The constraint is the circle of radius 1 centred at $(1, 1)$. Since u is increasing in the product $\sigma_1 \sigma_2$, its isoquants are hyperbolas $\sigma_1 \sigma_2 = k^2$.

Final answer

$$\boxed{\mathcal{L} = \sqrt{\sigma_1 \sigma_2} - \lambda \left((\sigma_1 - 1)^2 + (\sigma_2 - 1)^2 - 1 \right).}$$

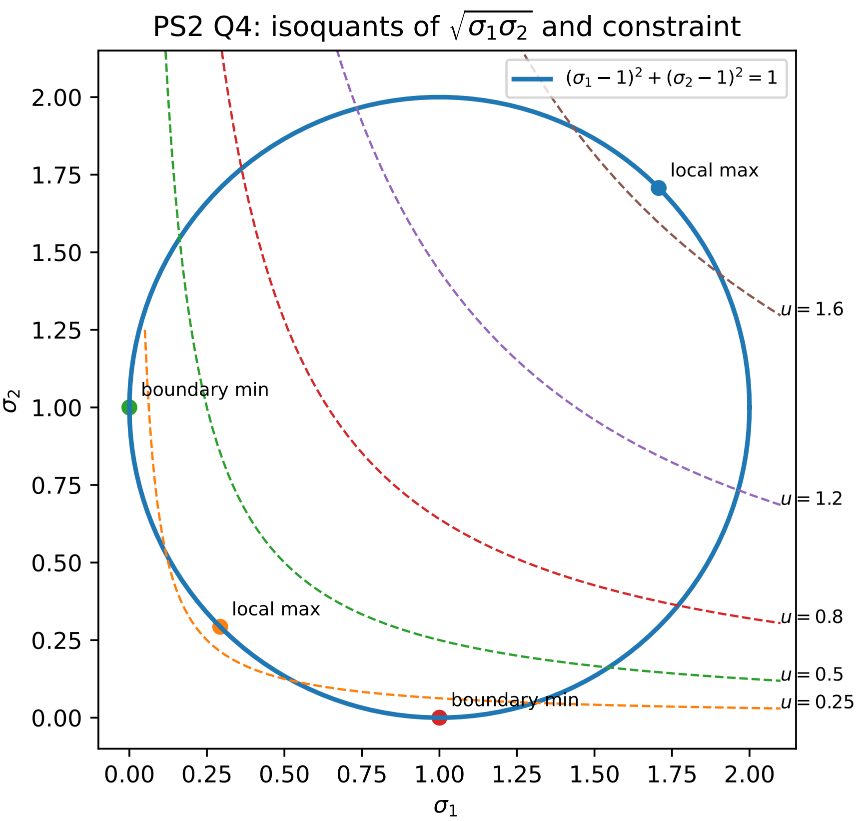

Figure 2: Constraint circle and isoquants for $u(\boldsymbol{\sigma}) = \sqrt{\sigma_1\sigma_2}$. With strict positivity, the axis points are excluded. If non-negativity is allowed, they become boundary minima.

4(b)

Question in standard notation

Find the first-order conditions and all local optima for $\boldsymbol{\sigma} > \mathbf{0}$.

The derivatives of u are

$$\frac{\partial u}{\partial \sigma_1} = \frac{1}{2}\sqrt{\frac{\sigma_2}{\sigma_1}}, \qquad \frac{\partial u}{\partial \sigma_2} = \frac{1}{2}\sqrt{\frac{\sigma_1}{\sigma_2}}.$$

The first-order conditions are

$$\begin{aligned} \frac{1}{2}\sqrt{\frac{\sigma_2}{\sigma_1}} - 2\lambda(\sigma_1 - 1) &= 0, \\ \frac{1}{2}\sqrt{\frac{\sigma_1}{\sigma_2}} - 2\lambda(\sigma_2 - 1) &= 0, \\ (\sigma_1 - 1)^2 + (\sigma_2 - 1)^2 &= 1. \end{aligned}$$

Dividing the first two equations gives

$$\frac{\sigma_2}{\sigma_1} = \frac{\sigma_1 - 1}{\sigma_2 - 1} \qquad \Longleftrightarrow \qquad (\sigma_2 - \sigma_1)(\sigma_1 + \sigma_2 - 1) = 0.$$

The second possibility, $\sigma_1 + \sigma_2 = 1$, meets the circle only at $(0, 1)$ and $(1, 0)$, which are excluded by $\boldsymbol{\sigma} > \mathbf{0}$. Hence $\sigma_1 = \sigma_2$. The constraint gives

$$2(\sigma_1 - 1)^2 = 1,$$

so

$$\sigma_1 = \sigma_2 = 1 \pm \frac{1}{\sqrt{2}}.$$

The corresponding multipliers satisfy $1/2 = 2\lambda(\sigma_1 - 1)$, hence

$$\lambda = \frac{1}{4(\sigma_1 - 1)} = \pm\frac{\sqrt{2}}{4}.$$

Final answer

$$\boxed{(\lambda^*, \sigma_1^*, \sigma_2^*) = \left(\frac{\sqrt{2}}{4}, 1 + \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}\right), \quad \left(-\frac{\sqrt{2}}{4}, 1 - \frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}\right)}.$$

$$\boxed{\text{Number of smooth local optima with } \boldsymbol{\sigma} > \mathbf{0} : 2.}$$

4(c)

Question in standard notation

Calculate the bordered Hessian and determine the nature of the local optima.

The Hessian of u is

$$\nabla^2 u = \begin{pmatrix} -\frac{1}{4}\frac{\sqrt{\sigma_2}}{\sigma_1^{3/2}} & \frac{1}{4\sqrt{\sigma_1\sigma_2}} \\ \frac{1}{4\sqrt{\sigma_1\sigma_2}} & -\frac{1}{4}\frac{\sqrt{\sigma_1}}{\sigma_2^{3/2}} \end{pmatrix}.$$

Since $\nabla^2 g = 2I_2$,

$$\nabla^2_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\mathcal{L} = \nabla^2 u - 2\lambda I_2.$$

The bordered Hessian is

$$H_B = \begin{pmatrix} 0 & 2(\sigma_1 - 1) & 2(\sigma_2 - 1) \\ 2(\sigma_1 - 1) & \mathcal{L}_{11} & \mathcal{L}_{12} \\ 2(\sigma_2 - 1) & \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}.$$

For classification, use a tangent vector. At both diagonal candidates the constraint gradient is proportional to $(1, 1)^\top$, so a tangent vector is $\mathbf{z} = (1, -1)^\top$. Direct substitution gives

$$\mathbf{z}^\top \nabla^2_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\mathcal{L}\,\mathbf{z} < 0$$

at both candidates. Therefore both are constrained local maxima.

Final answer

$$\boxed{H_B = \begin{pmatrix} 0 & 2(\sigma_1 - 1) & 2(\sigma_2 - 1) \\ 2(\sigma_1 - 1) & u_{11} - 2\lambda & u_{12} \\ 2(\sigma_2 - 1) & u_{21} & u_{22} - 2\lambda \end{pmatrix}.}$$

$$\boxed{\text{Both smooth local optima are local maxima.}}$$

4(d)

Question in standard notation

If the choice set is changed from $\boldsymbol{\sigma} > \mathbf{0}$ to $\boldsymbol{\sigma} \geq \mathbf{0}$, how does the number and nature of local optima change?

The circle also contains the axis points $(0, 1)$ and $(1, 0)$. They were excluded by strict positivity. If non-negativity is allowed, they become feasible. At both points,

$$u(\boldsymbol{\sigma}) = \sqrt{\sigma_1 \sigma_2} = 0.$$

Since $u \geq 0$ everywhere on the feasible set, both are global minima. The derivative of u is singular at the axes, so these points are boundary optima rather than smooth Lagrange candidates.

Final answer

$$\boxed{\text{Additional feasible optima: } (0, 1) \text{ and } (1, 0).}$$

$$\boxed{\text{They are boundary global minima.}}$$

2.5 Question 5: Differential equations

5(a)

Question in standard notation

Consider

$$\ddot{x}(t) + a\dot{x}(t) + \frac{b}{4}x(t) = 1.$$

Let $y(t) = \dot{x}(t)$ and rewrite the equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = M \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}.$$

Since $y = \dot{x}$, the first equation is $\dot{x} = y$. The original equation gives

$$\dot{y} = \ddot{x} = 1 - ay - \frac{b}{4}x.$$

Therefore

Final answer

$$\boxed{M = \begin{pmatrix} 0 & 1 \\ -b/4 & -a \end{pmatrix}, \qquad \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.}$$

5(b)

Question in standard notation

Find a stationary state and a condition on a, b for strict stability.

A stationary state satisfies $\dot{x} = 0$ and $\dot{y} = 0$. Hence $y^* = 0$ and

$$0 = 1 - \frac{b}{4}x^*,$$

so $x^* = 4/b$ when $b \neq 0$.

The eigenvalues of M solve

$$\det\begin{pmatrix} -r & 1 \\ -b/4 & -a-r \end{pmatrix} = r^2 + ar + \frac{b}{4} = 0.$$

For a two-dimensional continuous-time system, both eigenvalues have negative real parts exactly when the trace is negative and the determinant is positive. Here

$$\operatorname{tr}(M) = -a, \qquad \det(M) = \frac{b}{4}.$$

Thus strict stability requires $a > 0$ and $b > 0$.

Final answer

$$\boxed{\mathbf{x}^* = \begin{pmatrix} 4/b \\ 0 \end{pmatrix}, \qquad \text{strict stability iff } a > 0,\ b > 0.}$$

5(c)

Question in standard notation

If $b = -1$, find a solution $\mathbf{x}(t) = (x(t), y(t))^\top$ satisfying $x(0) = x_0$ and $y(0) = \dot{x}(0) = y_0$.

With $b = -1$,

$$\ddot{x} + a\dot{x} - \frac{1}{4}x = 1.$$

A constant particular solution is $x_p = -4$. The homogeneous characteristic equation is

$$r^2 + ar - \frac{1}{4} = 0,$$

with roots

$$r_+ = \frac{-a + \sqrt{a^2+1}}{2}, \qquad r_- = \frac{-a - \sqrt{a^2+1}}{2}.$$

Let $\Delta = \sqrt{a^2+1}$ and $z_0 = x_0 + 4$. Then

$$x(t) = -4 + C_+ e^{r_+ t} + C_- e^{r_- t}.$$

The initial conditions give

$$C_+ + C_- = z_0, \qquad r_+ C_+ + r_- C_- = y_0.$$

Solving,

$$C_+ = \frac{y_0 - r_- z_0}{\Delta}, \qquad C_- = \frac{r_+ z_0 - y_0}{\Delta}.$$

Finally $y(t) = \dot{x}(t)$.

Final answer

$$\boxed{\begin{aligned} x(t) &= -4 + \frac{y_0 - r_-(x_0+4)}{\Delta} e^{r_+ t} + \frac{r_+(x_0+4) - y_0}{\Delta} e^{r_- t}, \\ y(t) &= r_+ \frac{y_0 - r_-(x_0+4)}{\Delta} e^{r_+ t} + r_- \frac{r_+(x_0+4) - y_0}{\Delta} e^{r_- t}, \end{aligned}}$$

where $\Delta = \sqrt{a^2+1}$ and $r_\pm = (-a \pm \Delta)/2$.

5(d)

Question in standard notation

Show that the second-order equation is the first-order condition for a dynamic optimization problem with utility rate

$$u(x,\dot{x}) = \alpha x + \beta x^2 + \gamma \dot{x}^2,$$

and determine α, β, γ.

The Euler–Lagrange equation is

$$\frac{\partial u}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial u}{\partial \dot{x}}\right) = 0.$$

Here

$$\frac{\partial u}{\partial x} = \alpha + 2\beta x, \qquad \frac{\partial u}{\partial \dot{x}} = 2\gamma \dot{x}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial u}{\partial \dot{x}}\right) = 2\gamma \ddot{x}.$$

Therefore

$$\alpha + 2\beta x - 2\gamma \ddot{x} = 0 \quad \Longleftrightarrow \quad \ddot{x} - \frac{\beta}{\gamma} x = \frac{\alpha}{2\gamma}.$$

This equation has no $\dot{x}$ term. Therefore the displayed utility rate matches the given differential equation exactly only when $a = 0$. In that case we need

$$-\frac{\beta}{\gamma} = \frac{b}{4}, \qquad \frac{\alpha}{2\gamma} = 1.$$

The scale of (α, β, γ) is not unique. Taking $\gamma = 1$ gives

Final answer

$$\boxed{\text{Exact match under ordinary Euler–Lagrange requires } a = 0.}$$

$$\boxed{\gamma = 1, \qquad \alpha = 2, \qquad \beta = -\frac{b}{4}.}$$

If the intended problem included an explicit time weight such as $\mathrm{e}^{at}u(x,\dot{x})$, a first-derivative term would appear. That extra assumption is not stated in the question.

3 Solutions to ECON0010 Late Summer Assessment 2025

3.1 Question 1: Differential equations and opinion dynamics

1(a)

Question in standard notation

Two agents have opinions $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)^\top$ and evidence $\boldsymbol{\eta} = (\eta_1, \eta_2)^\top$. Agent j has utility

$$u_j(\boldsymbol{\sigma}) = \eta_j \sigma_j - \frac{1}{2}\sigma_j (R\boldsymbol{\sigma})_j, \qquad R = \begin{pmatrix} R_1 & -C_{12} \\ -C_{21} & R_2 \end{pmatrix},$$

with $R_j > 0$ and $C_{jk} > 0$. Each agent chooses only their own opinion. Find the first-order conditions and the matrix M such that $\boldsymbol{\eta} = M\boldsymbol{\sigma}^*$.

For agent 1,

$$(R\boldsymbol{\sigma})_1 = R_1\sigma_1 - C_{12}\sigma_2,$$

so

$$u_1 = \eta_1\sigma_1 - \frac{1}{2}R_1\sigma_1^2 + \frac{1}{2}C_{12}\sigma_1\sigma_2.$$

The first-order condition with respect to σ_1 is

$$\eta_1 - R_1\sigma_1 + \frac{1}{2}C_{12}\sigma_2 = 0.$$

Similarly,

$$\eta_2 - R_2\sigma_2 + \frac{1}{2}C_{21}\sigma_1 = 0.$$

Rearranging,

$$\eta_1 = R_1\sigma_1 - \frac{1}{2}C_{12}\sigma_2, \qquad \eta_2 = -\frac{1}{2}C_{21}\sigma_1 + R_2\sigma_2.$$

Final answer

$$\boxed{\boldsymbol{\eta} = M\boldsymbol{\sigma}^*, \qquad M = \begin{pmatrix} R_1 & -C_{12}/2 \\ -C_{21}/2 & R_2 \end{pmatrix}.}$$

1(b)

Question in standard notation

Assume $R_1 = R_2 = 1$, $C_{12} = C_{21} = 2/c$ with $c > 0$, and continuous-time gradient learning with learning rate γ. Find the differential equations, the stationary state, and the range of c for stability.

Under these assumptions,

$$\frac{1}{2}C_{12} = \frac{1}{2}C_{21} = \frac{1}{c}.$$

Gradient learning means

$$\dot{\sigma}_j(t) = \gamma\frac{\partial u_j}{\partial \sigma_j}.$$

Therefore

$$\dot{\sigma}_1 = \gamma\left(\eta_1 - \sigma_1 + \frac{1}{c}\sigma_2\right), \qquad \dot{\sigma}_2 = \gamma\left(\eta_2 - \sigma_2 + \frac{1}{c}\sigma_1\right).$$

In vector form,

$$\dot{\boldsymbol{\sigma}} = \gamma(\boldsymbol{\eta} - M_0\boldsymbol{\sigma}), \qquad M_0 = \begin{pmatrix} 1 & -1/c \\ -1/c & 1 \end{pmatrix}.$$

The stationary state solves $M_0\boldsymbol{\sigma}_s = \boldsymbol{\eta}$. Since

$$M_0^{-1} = \frac{1}{1-c^{-2}} \begin{pmatrix} 1 & 1/c \\ 1/c & 1 \end{pmatrix},$$

we obtain

$$\boldsymbol{\sigma}_s = \frac{1}{c^2-1} \begin{pmatrix} c^2\eta_1 + c\eta_2 \\ c\eta_1 + c^2\eta_2 \end{pmatrix}.$$

For stability, deviations satisfy

$$\dot{\mathbf{z}} = \gamma N \mathbf{z}, \qquad N = \begin{pmatrix} -1 & 1/c \\ 1/c & -1 \end{pmatrix}.$$

The eigenvalues of N are

$$-1 + \frac{1}{c}, \qquad -1 - \frac{1}{c}.$$

Since $\gamma > 0$ and $c > 0$, both are negative exactly when $c > 1$.

Final answer

$$\boxed{\dot{\sigma}_1 = \gamma\left(\eta_1 - \sigma_1 + \frac{\sigma_2}{c}\right), \qquad \dot{\sigma}_2 = \gamma\left(\eta_2 - \sigma_2 + \frac{\sigma_1}{c}\right).}$$

$$\boxed{\boldsymbol{\sigma}_s = \frac{1}{c^2-1} \begin{pmatrix} c^2\eta_1 + c\eta_2 \\ c\eta_1 + c^2\eta_2 \end{pmatrix}.}$$

$$\boxed{\text{Strict stability iff } c > 1.}$$

1(c)

Question in standard notation

Show that the general solution is

$$\boldsymbol{\sigma}(t) = \mathrm{e}^{\gamma N t}(\boldsymbol{\sigma}(0) - \boldsymbol{\sigma}_s) + \boldsymbol{\sigma}_s,$$

for the constant dynamic matrix N, and calculate $\mathrm{e}^{\gamma N t}$.

From part (b),

$$\dot{\boldsymbol{\sigma}} = \gamma(\boldsymbol{\eta} - M_0\boldsymbol{\sigma}).$$

Since $M_0\boldsymbol{\sigma}_s = \boldsymbol{\eta}$, subtracting the stationary state gives

$$\dot{\boldsymbol{\sigma}} = \gamma M_0(\boldsymbol{\sigma}_s - \boldsymbol{\sigma}) = -\gamma M_0(\boldsymbol{\sigma} - \boldsymbol{\sigma}_s).$$

Thus $N = -M_0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_s) = \gamma N(\boldsymbol{\sigma} - \boldsymbol{\sigma}_s).$$

This has solution

$$\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_s = \mathrm{e}^{\gamma N t}(\boldsymbol{\sigma}(0) - \boldsymbol{\sigma}_s).$$

The matrix

$$N = \begin{pmatrix} -1 & 1/c \\ 1/c & -1 \end{pmatrix}$$

has eigenvectors $(1,1)^\top$ and $(1,-1)^\top$, with eigenvalues $-(1-1/c)$ and $-(1+1/c)$. Hence

$$E_+(t) = \mathrm{e}^{-\gamma(1-1/c)t}, \qquad E_-(t) = \mathrm{e}^{-\gamma(1+1/c)t},$$

and

$$\mathrm{e}^{\gamma N t} = \frac{1}{2}\begin{pmatrix} E_+(t)+E_-(t) & E_+(t)-E_-(t) \\ E_+(t)-E_-(t) & E_+(t)+E_-(t) \end{pmatrix}.$$

Final answer

$$\boxed{\boldsymbol{\sigma}(t) = \mathrm{e}^{\gamma N t}(\boldsymbol{\sigma}(0) - \boldsymbol{\sigma}_s) + \boldsymbol{\sigma}_s.}$$

$$\boxed{\mathrm{e}^{\gamma N t} = \frac{1}{2}\begin{pmatrix} E_+ + E_- & E_+ - E_- \\ E_+ - E_- & E_+ + E_- \end{pmatrix}, \quad E_\pm = \mathrm{e}^{-\gamma(1\mp 1/c)t}.}$$

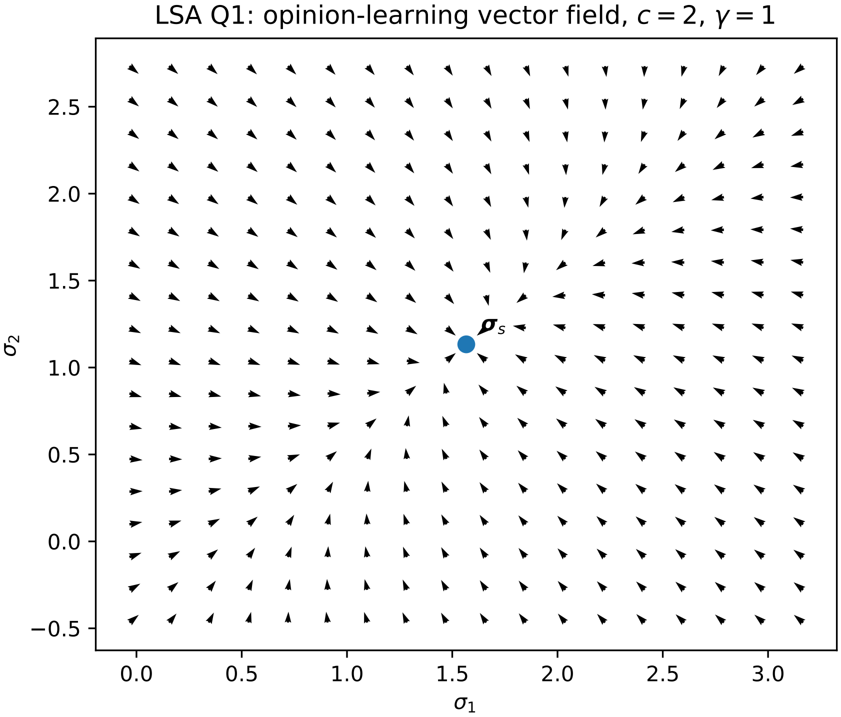

Figure 3: Illustrative opinion-learning vector field for LSA Question 1 with $c = 2$ and $\gamma = 1$.

1(d)

Question in standard notation

Find the marginal utility of c for both agents at the stable stationary state.

With $C_{12} = C_{21} = 2/c$, agent 1's utility is

$$u_1 = \eta_1\sigma_1 - \frac{1}{2}\sigma_1^2 + \frac{1}{c}\sigma_1\sigma_2.$$

By the envelope theorem, the effect of c through agent 1's own optimized choice does not need to be included. Holding opinions fixed,

$$\frac{\partial u_1}{\partial c} = -\frac{1}{c^2}\sigma_1\sigma_2.$$

The same expression holds for agent 2. At the stationary state,

$$\sigma_{1s} = \frac{c(c\eta_1 + \eta_2)}{c^2 - 1}, \qquad \sigma_{2s} = \frac{c(\eta_1 + c\eta_2)}{c^2 - 1}.$$

Therefore

$$-\frac{\sigma_{1s}\sigma_{2s}}{c^2} = -\frac{(c\eta_1 + \eta_2)(\eta_1 + c\eta_2)}{(c^2 - 1)^2}.$$

Final answer

$$\boxed{\frac{\partial u_1}{\partial c} = \frac{\partial u_2}{\partial c} = -\frac{(c\eta_1 + \eta_2)(\eta_1 + c\eta_2)}{(c^2 - 1)^2}.}$$

The negative sign is natural when the stationary opinions have the same sign: increasing c reduces the complementarity coefficient $1/c$.

3.2 Question 2: Constrained optimization

2(a)

Question in standard notation

Maximize or optimize

$$f(\mathbf{x}) = x_1x_2x_3$$

subject to

$$g(\mathbf{x}) = \mathbf{p}^\top\mathbf{x} - 15 = 0, \qquad \mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 1/2 \end{pmatrix}.$$

Write the Lagrangian.

Using $\mathcal{L} = f - \lambda g$,

Final answer

$$\boxed{\mathcal{L} = x_1x_2x_3 - \lambda\left(x_1 + 2x_2 + \frac{1}{2}x_3 - 15\right).}$$

2(b)

Question in standard notation

Find the first-order conditions and all local optima $\mathbf{x}^*$ with $x_j^* \geq 0$ for $j = 1, 2, 3$.

The equality-Lagrange first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1} = x_2x_3 - \lambda = 0,$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = x_1x_3 - 2\lambda = 0,$$
$$\frac{\partial \mathcal{L}}{\partial x_3} = x_1x_2 - \frac{1}{2}\lambda = 0,$$
$$x_1 + 2x_2 + \frac{1}{2}x_3 = 15.$$

For an interior solution, $x_1, x_2, x_3 > 0$. Dividing the first two equations gives $x_1 = 2x_2$. Comparing the first and third gives $x_3 = 2x_1$. Thus $x_3 = 4x_2$. The constraint becomes

$$2x_2 + 2x_2 + 2x_2 = 15,$$

so

$$x_2 = \frac{5}{2}, \qquad x_1 = 5, \qquad x_3 = 10.$$

The multiplier is $\lambda = x_2x_3 = 25$.

The equality-Lagrange equations also allow the intercept candidates with two variables zero. These are

$$(0, 0, 30), \quad (0, 15/2, 0), \quad (15, 0, 0),$$

all with $\lambda = 0$.

Final answer

$$\boxed{\begin{gathered} x_2x_3 - \lambda = 0, \quad x_1x_3 - 2\lambda = 0, \quad x_1x_2 - \frac{1}{2}\lambda = 0, \\ x_1 + 2x_2 + \frac{1}{2}x_3 - 15 = 0. \end{gathered}}$$

$$\boxed{(0, 0, 30; 0), \quad (0, 15/2, 0; 0), \quad (5, 5/2, 10; 25), \quad (15, 0, 0; 0),}$$

where the fourth entry is λ.

Ambiguity or interpretation

If $x_j \geq 0$ is treated as a full inequality constraint, then every feasible point with at least one zero component has $f = 0$ and is a weak global minimum because $f \geq 0$ on the non-negative budget simplex. The four points above are the candidates obtained by the displayed equality-Lagrange equations.

2(c)

Question in standard notation

Determine the bordered Hessian and evaluate the nature of all local optima.

The Hessian of $f = x_1x_2x_3$ is

$$H_f = \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}.$$

Since the constraint is linear, $\nabla^2 g = 0$, so this is also the Hessian of the Lagrangian with respect to $\mathbf{x}$. With the convention that the constraint gradient is placed in the first row and column,

$$H_B = \begin{pmatrix} 0 & 1 & 2 & 1/2 \\ 1 & 0 & x_3 & x_2 \\ 2 & x_3 & 0 & x_1 \\ 1/2 & x_2 & x_1 & 0 \end{pmatrix}.$$

At the interior candidate $(5, 5/2, 10)$, tangent directions satisfy

$$z_1 + 2z_2 + \frac{1}{2}z_3 = 0.$$

Writing $z_1 = -2z_2 - z_3/2$, the restricted quadratic form is

$$\mathbf{z}^\top H_f \mathbf{z} = -\frac{5}{2}\left(16z_2^2 + 4z_2z_3 + z_3^2\right) < 0$$

for any nonzero tangent vector. Therefore the interior candidate is a strict constrained local maximum. In fact it is the unique global maximum for the non-negative budget simplex.

At the intercepts, if negative values were allowed they would be saddle-type for the equality problem. Under the non-negativity restriction, they are weak boundary minima with value 0.

Final answer

$$\boxed{H_B = \begin{pmatrix} 0 & 1 & 2 & 1/2 \\ 1 & 0 & x_3 & x_2 \\ 2 & x_3 & 0 & x_1 \\ 1/2 & x_2 & x_1 & 0 \end{pmatrix}}.$$

$$\boxed{\begin{array}{c} (5, 5/2, 10) \text{ is the unique global maximum;} \\ \text{boundary zero-product points are weak minima under } x_j \geq 0. \end{array}}$$

2(d)

Question in standard notation

Explain the interpretation of the value found for the Lagrange multiplier.

For the interior maximum, $\lambda^* = 25$ under the Lagrangian

$$\mathcal{L} = f - \lambda(p^\top x - 15).$$

If the right-hand side of the constraint is changed from 15 to B, the value function is

$$V(B) = \max\{x_1x_2x_3 : x_1 + 2x_2 + x_3/2 = B,\ x_j \geq 0\}.$$

The envelope theorem gives

$$V'(15) = \lambda^* = 25.$$

So one additional unit of the resource/budget raises the optimized objective by approximately 25 units near $B = 15$.

Final answer

$$\boxed{\lambda^* = 25 \text{ is the marginal value of relaxing the constraint } p^\top x = 15.}$$

3.3 Question 3: Difference equations

3(a)

Question in standard notation

Consider

$$x_{t+1} - x_t = -ax_t + by_t + x_0, \qquad y_{t+1} - y_t = -ay_t + bx_t + y_0.$$

Write this as $\mathbf{x}_{t+1} - \mathbf{x}_t = M\mathbf{x}_t + \mathbf{x}_0$.

Let $\mathbf{x}_t = (x_t, y_t)^\top$ and $\mathbf{x}_0 = (x_0, y_0)^\top$. Then

$$M = \begin{pmatrix} -a & b \\ b & -a \end{pmatrix}.$$

Final answer

$$\boxed{M = \begin{pmatrix} -a & b \\ b & -a \end{pmatrix}, \qquad \mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.}$$

3(b)

Question in standard notation

Calculate the stationary state $\mathbf{x}_s$.

A stationary state solves

$$\mathbf{0} = M\mathbf{x}_s + \mathbf{x}_0,$$

so

$$M\mathbf{x}_s = -\mathbf{x}_0.$$

Equivalently,

$$\begin{pmatrix} a & -b \\ -b & a \end{pmatrix} \begin{pmatrix} x_s \\ y_s \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

The determinant is $a^2 - b^2$, and the inverse is

$$\frac{1}{a^2 - b^2} \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Thus

Final answer

$$\boxed{\mathbf{x}_s = \begin{pmatrix} \dfrac{ax_0 + by_0}{a^2 - b^2} \\ \dfrac{bx_0 + ay_0}{a^2 - b^2} \end{pmatrix}, \qquad a^2 \neq b^2.}$$

3(c)

Question in standard notation

Determine the conditions on $a, b > 0$ for stability, sketch the region in the (a, b) plane, and give the general solution in terms of $\mathbf{x}_0$, $\mathbf{x}_s$, and a time-dependent matrix B_t.

Deviations $\mathbf{z}_t = \mathbf{x}_t - \mathbf{x}_s$ satisfy

$$\mathbf{z}_{t+1} = (I + M)\mathbf{z}_t, \qquad I + M = \begin{pmatrix} 1 - a & b \\ b & 1 - a \end{pmatrix}.$$

The eigenvectors are $(1, 1)^\top$ and $(1, -1)^\top$, with eigenvalues

$$\mu_+ = 1 - a + b, \qquad \mu_- = 1 - a - b.$$

Strict stability requires

$$|1-a+b|<1, \qquad |1-a-b|<1.$$

For $a, b > 0$, this is equivalent to

$$a>b, \qquad a+b<2.$$

The general solution is

$$\mathbf{x}_t = B_t(\mathbf{x}_0 - \mathbf{x}_s) + \mathbf{x}_s,$$

where

$$B_t = (I+M)^t = \frac{1}{2}\begin{pmatrix} \mu_+^t + \mu_-^t & \mu_+^t - \mu_-^t \\ \mu_+^t - \mu_-^t & \mu_+^t + \mu_-^t \end{pmatrix}.$$

Final answer

$$\boxed{|1-a+b|<1, \qquad |1-a-b|<1.}$$

$$\boxed{\text{For } a,b>0: \quad b<a<2-b.}$$

$$\boxed{B_t = \frac{1}{2}\begin{pmatrix} (1-a+b)^t + (1-a-b)^t & (1-a+b)^t - (1-a-b)^t \\ (1-a+b)^t - (1-a-b)^t & (1-a+b)^t + (1-a-b)^t \end{pmatrix}.}$$

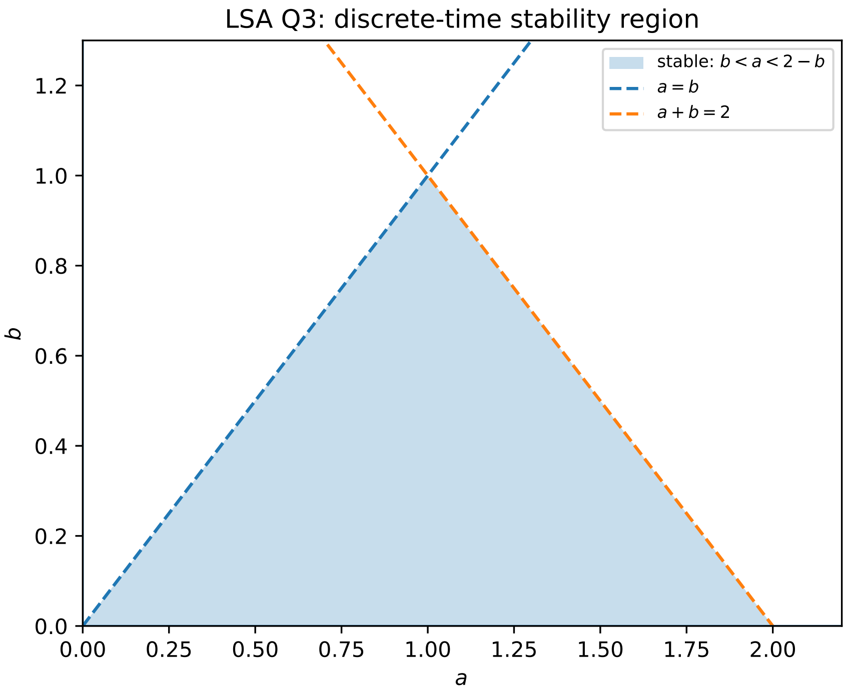

Figure 4: Stability region for LSA Question 3: $a, b > 0$, $a > b$, and $a + b < 2$.

3(d)

Question in standard notation

The paper displays a perturbed system with $|\delta| \ll 1$:

$$\dot{x}(t) = -ax(t) + b(1+\delta)y(t) + x_0, \qquad \dot{y}(t) = -ay(t) + b(1-\delta)x(t) + y_0.$$

Find the new M, calculate the eigenvalues, and give quadratic approximations of the eigenvalues and eigenvectors. Comment.

The notation changes from a difference equation to $\dot{x}$ in the paper. The coefficient matrix is the same object to diagonalize in either case; for the discrete-time stability matrix one would add I.

The new coefficient matrix is

$$M_\delta = \begin{pmatrix} -a & b(1+\delta) \\ b(1-\delta) & -a \end{pmatrix}.$$

Its eigenvalues solve

$$(-a-m)^2 - b^2(1-\delta^2) = 0,$$

so

$$m_+ = -a + b\sqrt{1-\delta^2}, \qquad m_- = -a - b\sqrt{1-\delta^2}.$$

Using $\sqrt{1-\delta^2} = 1 - \delta^2/2 + O(\delta^4)$,

$$m_+ = -a + b - \frac{b}{2}\delta^2 + O(\delta^4), \qquad m_- = -a - b + \frac{b}{2}\delta^2 + O(\delta^4).$$

Because M_δ is not symmetric when $\delta \neq 0$, right and left eigenvectors differ. A convenient exact choice is

$$\mathbf{r}_+ = \begin{pmatrix} \sqrt{1+\delta} \\ \sqrt{1-\delta} \end{pmatrix}, \quad \mathbf{r}_- = \begin{pmatrix} \sqrt{1+\delta} \\ -\sqrt{1-\delta} \end{pmatrix},$$

for right eigenvectors, and

$$\ell_+ = \begin{pmatrix} \sqrt{1-\delta} \\ \sqrt{1+\delta} \end{pmatrix}, \quad \ell_- = \begin{pmatrix} \sqrt{1-\delta} \\ -\sqrt{1+\delta} \end{pmatrix}$$

for left eigenvectors. Since

$$\sqrt{1\pm\delta} = 1 \pm \frac{\delta}{2} - \frac{\delta^2}{8} + O(\delta^3),$$

the eigenvectors shift at first order in δ, while the eigenvalues shift only at second order.

Final answer

$$\boxed{M_\delta = \begin{pmatrix} -a & b(1+\delta) \\ b(1-\delta) & -a \end{pmatrix}.}$$

$$\boxed{m_+ \approx -a + b - \frac{b}{2}\delta^2, \qquad m_- \approx -a - b + \frac{b}{2}\delta^2.}$$

Asymmetry changes eigenvectors at order δ but eigenvalues at order δ^2.

3.4 Question 4: Matrix algebra

4(a)

Question in standard notation

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Calculate $\det(A)$, the characteristic equation for an eigenvalue m, and $\mathbf{x} = (x, y)^\top$.

The determinant is

$$\det(A) = ad - bc.$$

Assuming $ad - bc \neq 0$,

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d + b \\ -c - a \end{pmatrix}.$$

The characteristic equation is

$$\det(A - mI) = \det \begin{pmatrix} a - m & b \\ c & d - m \end{pmatrix} = 0,$$

so

$$(a - m)(d - m) - bc = 0 \iff m^2 - (a + d)m + (ad - bc) = 0.$$

Final answer

$$\boxed{\det(A) = ad - bc.}$$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} b + d \\ -(a + c) \end{pmatrix}.}$$

$$\boxed{m^2 - (a + d)m + (ad - bc) = 0.}$$

4(b)

Question in standard notation

If $\alpha \in \mathbb{R}$, show directly that $\det(\alpha A) = \alpha^2 \det(A)$. Comment on the $n \times n$ case.

We have

$$\alpha A = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}.$$

Therefore

$$\det(\alpha A) = (\alpha a)(\alpha d) - (\alpha b)(\alpha c) = \alpha^2 (ad - bc) = \alpha^2 \det(A).$$

For an $n \times n$ matrix, multiplying all entries by α multiplies each of the n rows by α, so the determinant is multiplied by α^n.

Final answer

$$\boxed{\det(\alpha A) = \alpha^2 \det(A) \quad \text{for } 2 \times 2 \text{ matrices.}}$$

$$\boxed{\det(\alpha A) = \alpha^n \det(A) \quad \text{for } n \times n \text{ matrices.}}$$

4(c)

Question in standard notation

Show that

$$\det(A)I - \operatorname{tr}(A)A + A^2 = 0.$$

This is the 2×2 Cayley–Hamilton identity. We can verify it explicitly. First,

$$\operatorname{tr}(A) = a + d, \qquad A^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}.$$

Then

$$\det(A)I - \operatorname{tr}(A)A + A^2$$
$$= \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} - \begin{pmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{pmatrix} + \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}.$$

Each entry simplifies to zero.

Final answer

$$\boxed{\det(A)I - \operatorname{tr}(A)A + A^2 = 0.}$$

4(d)

Question in standard notation

Assume $\det(A) \neq 0$. Decide whether the following statements are true:

$$\det(A) = \frac{1}{2}\left((\operatorname{tr} A)^2 - \operatorname{tr}(A^2)\right),$$

and

$$\det(M) = \det(\operatorname{tr}(M)I - M).$$

For the first statement,

$$\operatorname{tr}(A) = a + d,$$

and

$$A^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}, \quad \operatorname{tr}(A^2) = a^2 + 2bc + d^2.$$

Therefore

$$\frac{1}{2}\left((a+d)^2 - (a^2 + 2bc + d^2)\right) = ad - bc = \det(A).$$

So the first statement is true.

For any 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\operatorname{tr}(M)I - M = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

whose determinant is $ad - bc = \det(M)$. So the second statement is also true.

Final answer

$$\boxed{\det(A) = \frac{1}{2}\left((\operatorname{tr} A)^2 - \operatorname{tr}(A^2)\right) \text{ is true.}}$$

$$\boxed{\det(M) = \det(\operatorname{tr}(M)I - M) \text{ is true for } 2 \times 2 \text{ matrices.}}$$

3.5 Question 5: Optimization and dynamic optimization

5(a)

Question in standard notation

Three agents have opinions $\sigma_1, \sigma_2, \sigma_3$, common evidence η, and community utility

$$U(\boldsymbol{\sigma}) = \sum_{j=1}^{3} \left(\eta\sigma_j - \frac{R}{2}\sigma_j^2 - \frac{\mu}{2}\left((\sigma_j - \sigma_{j+1})^2 + (\sigma_{j-1} - \sigma_j)^2\right)\right),$$

where $\sigma_0 = \sigma_4 = 0$ and $\mu, R > 0$. Find the first-order conditions.

The penalty terms double-count neighbour disagreement between agents and also penalize the endpoints for being away from the outside fixed opinions 0. Differentiating with respect to σ_1, σ_2, and σ_3 gives

$$\eta - (R + 3\mu)\sigma_1 + 2\mu\sigma_2 = 0,$$

$$\eta + 2\mu\sigma_1 - (R + 4\mu)\sigma_2 + 2\mu\sigma_3 = 0,$$

$$\eta + 2\mu\sigma_2 - (R + 3\mu)\sigma_3 = 0.$$

Equivalently,

Final answer

$$\boxed{(R + 3\mu)\sigma_1 - 2\mu\sigma_2 = \eta,}$$

$$\boxed{-2\mu\sigma_1 + (R + 4\mu)\sigma_2 - 2\mu\sigma_3 = \eta,}$$

$$\boxed{-2\mu\sigma_2 + (R + 3\mu)\sigma_3 = \eta.}$$

5(b)

Question in standard notation

Solve the first-order conditions and interpret the result.

By symmetry, $\sigma_1 = \sigma_3 = s$ and $\sigma_2 = m$. The first and second equations become

$$(R + 3\mu)s - 2\mu m = \eta,$$

$$(R + 4\mu)m - 4\mu s = \eta.$$

Solving this 2×2 system gives

$$s = \frac{\eta(R + 6\mu)}{R^2 + 7R\mu + 4\mu^2}, \qquad m = \frac{\eta(R + 7\mu)}{R^2 + 7R\mu + 4\mu^2}.$$

Hence

Final answer

$$\boxed{\sigma_1^* = \frac{\eta(R + 6\mu)}{R^2 + 7R\mu + 4\mu^2}, \quad \sigma_2^* = \frac{\eta(R + 7\mu)}{R^2 + 7R\mu + 4\mu^2}, \quad \sigma_3^* = \frac{\eta(R + 6\mu)}{R^2 + 7R\mu + 4\mu^2}.}$$

The middle agent has the largest opinion when $\eta > 0$ because the endpoint agents are pulled toward the boundary opinions $\sigma_0 = \sigma_4 = 0$. If $\mu = 0$, all agents choose η/R, as they would without neighbour effects.

5(c)

Question in standard notation

Calculate the Hessian and classify the solution.

The Hessian of U with respect to $(\sigma_1, \sigma_2, \sigma_3)$ is

$$H = \begin{pmatrix} -(R + 3\mu) & 2\mu & 0 \\ 2\mu & -(R + 4\mu) & 2\mu \\ 0 & 2\mu & -(R + 3\mu) \end{pmatrix}.$$

To classify, examine $-H$:

$$-H = \begin{pmatrix} R + 3\mu & -2\mu & 0 \\ -2\mu & R + 4\mu & -2\mu \\ 0 & -2\mu & R + 3\mu \end{pmatrix}.$$

The leading principal minors are positive for $R, \mu > 0$:

$$R + 3\mu > 0,$$

$$(R + 3\mu)(R + 4\mu) - 4\mu^2 = R^2 + 7R\mu + 8\mu^2 > 0,$$

and the full determinant is also positive. Hence $-H$ is positive definite, so H is negative definite. The critical point is therefore the unique global maximum of this strictly concave quadratic objective.

Final answer

$$\boxed{H = \begin{pmatrix} -(R + 3\mu) & 2\mu & 0 \\ 2\mu & -(R + 4\mu) & 2\mu \\ 0 & 2\mu & -(R + 3\mu) \end{pmatrix}.}$$

The solution is a strict global maximum.

5(d)

Question in standard notation

For a continuum of agents $x \in [0,1]$ with opinion $\sigma(x)$, suppose community utility is

$$U[\sigma] = \int_0^1 \left(\eta\sigma(x) - \frac{R}{2}\sigma(x)^2 - \frac{\mu}{2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}x}\right)^2 \right) \mathrm{d}x.$$

Find the differential equation whose solution optimizes utility and state the type of differential equation.

The integrand is

$$F(\sigma, \sigma') = \eta\sigma - \frac{R}{2}\sigma^2 - \frac{\mu}{2}(\sigma')^2.$$

The Euler–Lagrange equation is

$$\frac{\partial F}{\partial \sigma} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \sigma'}\right) = 0.$$

Here

$$\frac{\partial F}{\partial \sigma} = \eta - R\sigma, \qquad \frac{\partial F}{\partial \sigma'} = -\mu\sigma'.$$

Therefore

$$\eta - R\sigma - \frac{\mathrm{d}}{\mathrm{d}x}(-\mu\sigma') = 0 \quad \Longleftrightarrow \quad \eta - R\sigma + \mu\sigma'' = 0.$$

Equivalently,

$$R\sigma(x) - \mu\sigma''(x) = \eta.$$

Final answer

$$\boxed{\mu\sigma''(x) - R\sigma(x) + \eta = 0.}$$

A linear, second-order, ordinary differential equation with constant coefficients.

The boundary conditions are not stated in the question. If the continuum analogue of $\sigma_0 = \sigma_4 = 0$ is imposed, one would use fixed boundary conditions such as $\sigma(0) = \sigma(1) = 0$.

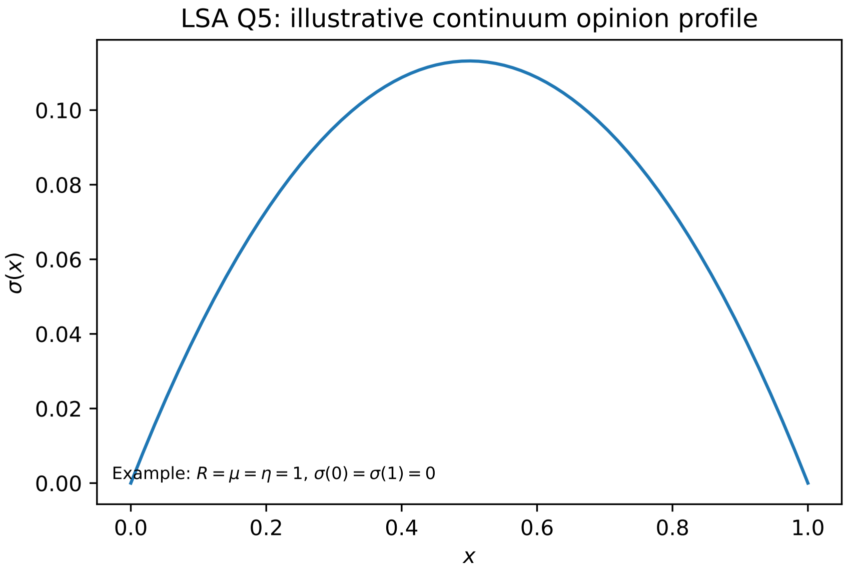

Figure 5: Illustrative continuum solution under fixed endpoints $\sigma(0) = \sigma(1) = 0$ for $R = \mu = \eta = 1$. The figure is illustrative because the exam question asks only for the Euler–Lagrange differential equation.